OPTIMALITY OF A STOP-LOSS REINSURANCE IN LAYERS

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Abstract.

We determine the optimal two-layers stop-loss reinsurance contracts, which minimize the total splitting risk measure under market conditions. Then, following a CAPM like pricing methodology from the author, we show how to obtain for a portfolio in life insurance the limiting optimal stop-loss contracts, which offer the greatest reduction in cost of capital.

Key words: reinsurance, comonotone property, perfect hedge, two-layers stop-loss contract, total splitting risk measure, p-norm principle, economic capital, cost of capital

1. Introduction.

The design of “optimal” (re)insurance contracts is an important actuarial subject, which in view of the great diversity of possible meaningful decision criteria, finds many different solutions. The present study is restricted to the “optimal” design of the two-layers stop-loss contract, whose choice is motivated by several results. The two-layers stop-loss contract has been shown “optimal” for any stop-loss order preserving criterion among the restricted set of reinsurance contracts generated by all stop-loss contracts in layers with fixed expected reinsurance net costs (e.g. Van Heerwaarden et al.(1989), Van Heerwaarden(1991), p. 121, Kaas et al.(1994), Example VIII.3.1, p. 86-87). It appears also as optimal reinsurance structure in the theory developed by Hesselager(1993). It belongs to the class of perfectly hedged all-finance derivative contracts introduced and studied by the author(1994/95/98). As most stop-loss like treaty it may serve as valuable substitute in situations a stop-loss contract is not available, undesirable or does not make sense (for this last point see Hürlimann(1993), Section 4, "Remarque"). Finally, it might be of interest to mention that best bounds for the expected value of the two-layers stop-loss contract by given range, mean and variance have been derived in Hürlimann(1997).

It is generally recognized that an appropriate reinsurance program should minimize unexpected fluctuations and produce value through more stable insurance results (e.g. Venter(2001)). Suppose unexpected fluctuations are measured using the variance, and that the unexpected aggregate fluctuations associated to the splitting of a risk in several components is measured using the total splitting risk measure defined by the sum of the component variances. Taking into account market conditions associated to a perfectly hedged contract with maximum deductible, we restrict the attention to those optimal reinsurance contracts, which minimize the corresponding total splitting risk measure.

To decide upon reinsurance, one often looks at the insolvency risk and the required economic capital to cover it. For a sound comparison of the price of reinsurance with the “cost of capital”, it is necessary to take into account the allocated economic capital that reinsurance is able to save (Corradin and Verbrigghe(2001)). Without reinsurance, the cost of capital (CoC) of an insurance risk is identified with the interest to be paid on the economic capital required for this insurance risk (net of the available initial capital). With a reinsurance, the
cost of capital identifies with the sum of the interest on the economic capital required for the retained insurance risk and the transfer of expected profit to the reinsurer. Using the CoC criterion, an optimal reinsurance contract is preferred to another provided its associated cost of capital is smaller.

A more detailed outline of the content follows. Section 2 reviews the class of perfectly hedged contracts with maximum deductible. Within this class, we formulate the optimization problem, which consists to minimize the total splitting risk measure under market conditions. Then, in Theorem 3.1, we derive the optimal two-layers stop-loss contracts, which solve our optimization problem. To set the price of reinsurance, we follow the pricing methodology introduced in Hürlimann(1994), which has been extended and generalized in Hürlimann(1998). Section 4 reviews this approach. In Section 5, the necessary background on economic capital and cost of capital is presented. The final Section 6 illustrates our findings for the optimal two-layers stop-loss contracts in the context of life insurance.

2. Optimal perfectly hedged contracts.

Given is a risk \( X \), that is a non-negative random variable with finite mean \( \mu \) and variance \( \sigma^2 \). An insurer is interested to keep the unexpected fluctuations arising from the coverage of \( X \) as small as possible in order to produce the most stable insurance results as possible. For this he seeks to split \( X \) by means of a risk-exchange or reinsurance in two smaller parts \( Y \) and \( Z \) such that \( X = Y + Z \). Suppose \( Z \) is the exchanged or reinsured part and \( Y \) the retained part. We assume that there exists non-decreasing functions \( f(x), g(x) \) such that \( 0 \leq f(x), g(x) \leq x \) and \( Y = f(X), Z = g(X) \). This means that the set of risk-exchanges or reinsurance contracts is restricted to those compensation functions for which neither contractual party will benefit in case the claim amount increases. Mathematically, \( Y \) and \( Z \) are called comonotonic random variables, a notion due to Schmeidler(1986) and Yaari(1987). The class of comonotonic random variables associated to \( X \) is denoted by \( \{ Y = f(X), Z = g(X) : Y + Z = X \} \).

Suppose unexpected fluctuations are measured using the variance. Then the unexpected aggregate fluctuations associated to a decomposition \( X = Y + Z \) with \( (Y, Z) \in Com(X) \) is measured by the total splitting risk measure defined by

\[
R_s[Y, Z] = \text{Var}[Y] + \text{Var}[Z] = \sigma^2 - 2 \cdot \text{Cov}[Y, Z].
\] (2.1)

From an economic viewpoint, it is desirable to maintain the unexpected aggregate fluctuations at a low level and thus minimize the quantity (2.1) respectively maximize the covariance between \( Y \) and \( Z \).

Hürlimann(1994/95/98) considers the important subclass of perfectly hedged contracts with maximum deductible \( d \) defined and denoted by

\[
S_d(X) = \{ (Y = f(X), Z = g(X)) \in Com(X) : d = \sup_{x \geq 0} \{ f(x) \} < \infty \}.
\] (2.2)

The attractiveness of this class stems from the following property. For \( (Y, Z) \in S_d(X) \) the non-negative function defined by \( d(x) = d - f(x), x \geq 0 \), defines a transformed random variable \( D = d(X) \) such that with probability one

\[
d + Z = X + D.
\] (2.3)
The amount \( d \) plus the covered part \( Z \) suffice to pay the claim \( X \) and there remains a random surplus amount \( D \geq 0 \). The latter quantity is interpreted as a \textit{perfectly hedged bonus} or dividend. In the following, let \( P \) denote the market price of a perfectly hedged contract \((Y, Z) \in S_d(X)\) with liability \( X + D \) and \( P_R \) the market price required to cover \( Z \). Then (2.3) implies the price relationship

\[
P = d + P_R.
\]

(2.4)

The above minimization of the total splitting risk measure leads to the following constrained optimization problem associated to a parametric family \((Y, Z) \in S_d(X)\):

\[
\text{max } \text{Cov}[Y, Z] \text{, under the conditions}
\]

\begin{align*}
(C1) & \quad E[Z] = c \\
(C2) & \quad P = d + P_R \\
(C3) & \quad d \geq \mu
\end{align*}

(2.5)

Let us motivate the considered conditions. Suppose market prices are set according to the expected value principle such that \( P = (1 + \theta) \cdot \mu \), \( P_R = (1 + \theta_R) \cdot E[Z] \), with \( \theta \neq \theta_R \) different \textit{loading factors}. Then (C1) says that the amount available for buying reinsurance is fixed. The condition (C2) is the price relationship (2.4). The condition (C3) means that the retained premium of the cedant, also denoted \( P_C = P - P_R = d \), should at least be equal to the expected value of the risk. Alternatively, the condition (C3) can be interpreted as a mean self-financing property of a self-reinsurance strategy, as explained in Hürlimann(1999), Section 2.

Since there are two equality constraints, to obtain a non-trivial solution of (2.5), the parametric family \((Y, Z) \in S_d(X)\) should at least contain three parameters. As motivated in the introduction, a meaningful candidate is the linear combination of stop-loss contracts in two layers

\[
Z = r(X - L)_+ + (1 - r)(X - M)_+ , \quad r \in (0,1), \quad M \geq L,
\]

(2.6)

whose maximum deductible is \( d = M - r(M - L) \).

3. \textbf{Optimal two-layers stop-loss contracts.}

The following result determines the two-layers stop-loss contracts, which define optimal perfectly hedged contracts in the sense of (2.5).

\textbf{Theorem 3.1.} Suppose \( X \) is a risk with finite mean \( \mu \). Let \((Y, Z) \in S_d(X)\) be defined by \( Z = r(X - L)_+ + (1 - r)(X - M)_+ \), \( r \in (0,1) \), \( M > L \), \( d = rL + (1 - r)M \). Then the constrained optimization problem \text{Cov}[Y, Z] = \text{max.} \) under the conditions

\begin{align*}
(C1) & \quad E[Z] = c \\
(C2) & \quad d = P - P_R \\
(C3) & \quad d \geq \mu
\end{align*}
has a maximum at $r = \frac{1}{2}$, $L$ is the unique solution of the equation $
abla(L) + \nabla(2d - L) = 2c$, where $d = P - P_r \geq \mu$, $M = 2d - L$, and $\nabla(x) = E[(X - x)_+]$ denotes the stop-loss transform of $X$.

**Proof.** For a random variable $X$, denote the distribution function by $F(x)$ and the survival function by $\bar{F}(x) = 1 - F(x)$. Denote the stop-loss transforms of degree 1 and 2 by $\nabla(x) = E[(X - x)_+]$ and $\nabla_2(x) = E[(X - x)_+^2]$, and set $\sigma^2(x) = Var[(X - x)_+] = \nabla_2(x) - \nabla(x)^2$.

For the given $(Y, Z)$ one has


with $D = rL - X + (1 - r)(M - X)_+$. By condition (C1) one has

$$E[D] = E[d - X + Z] = M - r(M - L) - \mu + c.$$  

Furthermore, one has

$$E[\bar{D}Z] = E[r^2(L - X)_+(X - L)_+ + r(1 - r)[(L - X)_+(X - M)_+ + (M - X)_+(X - L)_+]]$$

$$+ E[(1 - r)^2(X - M)_+(M - X)_+]$$

Since the first two and last of the summands of this expected value vanish, it remains the term

$$E[\bar{D}Z] = E[r(1 - r)(M - X)_+(X - L)_+]$$

Using the identity $(M - X)_+ = X + (X - M)_+$ and the relations

$$E[X(X - L)_+] = E[(X - L)_+^2 + L(X - L)_+] = \nabla_2(L) + L\nabla(L),$$

$$E[(X - M)_+(X - L)_+] = \bar{F}(M) \cdot E[(X - M)(X - M + M - L) | X > M],$$

one gets

$$E[\bar{D}Z] = r(1 - r) \cdot [\nabla_2(M) - \nabla_2(L) + (M - L)[\nabla(L) + \nabla(M)]]$$

With the above, this yields an expression for $Cov[Y, Z]$ in (3.1). To solve the constrained optimization problem, consider the Lagrange function

$$\Phi(r, L, M, \lambda, \nu) = Cov[Y, Z] - \lambda(c - E[Z]) - \nu(P - P_r - d)$$

$$= c[M - r(M - L) - \mu + c] + r(1 - r)[\nabla_2(L) - \nabla_2(M) - (M - L)[\nabla(L) + \nabla(M)]]$$

$$- \lambda[c - \nabla(M) - r[\nabla(L) - \nabla(M)]] - \nu[P - P_r - M + r(M - L)].$$

As a necessary condition for a maximum, the gradient of $\Phi$ must vanish. Using the derivatives $\nabla'(x) = -\bar{F}(x)$, $\nabla_2'(x) = -2\nabla(x)$, one obtains the conditions
\[ \Phi_r = -c(M - L) + (1 - 2r)[\pi_2(L) - \pi_2(M) - (M - L)[\pi(L) + \pi(M)] + \lambda[\pi(L) - \pi(M)] - \nu(M - L) = 0 \]  
\[ \Phi_L = cr + r(1 - r)[(M - L)\bar{F}(L) - [\pi(L) - \pi(M)]] - \lambda r\bar{F}(L) + vr = 0 \]  
\[ \Phi_M = c(1 - r) + r(1 - r)[(M - L)\bar{F}(M) - [\pi(L) - \pi(M)]] - \lambda(1 - r)\bar{F}(M) + \nu(1 - r) = 0 \]  
\[ \Phi_\nu = r[\pi(L) - \pi(M)] + \pi(M) - c = 0 \]  
\[ \Phi_v = P - P_r - M + r(M - L) = 0 \]

Since \( r \in (0,1) \) the conditions (L3) and (L2) yield the Lagrange multipliers

\[ \nu = \lambda\bar{F}(M) - c - r[(M - L)\bar{F}(M) - [\pi(L) - \pi(M)] \]  
\[ \lambda = \frac{(1 - 2r)[\pi(L) - \pi(M)] + (M - L)[r\bar{F}(M) - (1 - r)\bar{F}(L)]}{\bar{F}(M) - \bar{F}(L)} \]

Inserting these parameters into (L1), one obtains the equation between \( r, L, M \):

\[ (1 - 2r)[(M - L)[\pi(L) + \pi(M)] - [\pi_2(L) - \pi_2(M)]\{\bar{F}(M) - \bar{F}(L) \} = (1 - 2r)[(M - L)\bar{F}(L) - [\pi(L) - \pi(M)]]\{M - L)\bar{F}(M) - [\pi(L) - \pi(M)]\} \]

(3.8)

Unless \( r = \frac{1}{2} \) this equation has no solution with \( M > L \). To show this, consider the difference of the curly bracket products in (3.8):

\[ h(L,M) = [(M - L)[\pi(L) + \pi(M)] - [\pi_2(L) - \pi_2(M)]\{\bar{F}(M) - \bar{F}(L) \} - (M - L)\bar{F}(L) - [\pi(L) - \pi(M)]]\{M - L)\bar{F}(M) - [\pi(L) - \pi(M)]\} \]

(3.9)

Applying the mean value principle for integrals, there exists \( \xi, \eta \in (L,M) \) such that

\[ \pi(L) - \pi(M) = \int_L^M \bar{F}(x)dx = \bar{F}(\xi)(M - L), \]

(3.10)

\[ \pi_2(L) - \pi_2(M) = 2\int_L^M \pi(x)dx = 2\pi(\eta)(M - L). \]

(3.11)

Inserted into (3.9) one gets

\[ h(L,M) = (M - L)[\pi(\eta) - \pi(M)] - \pi(\eta) - \pi(M)]\{\bar{F}(M) - \bar{F}(L) \} \]

(3.12)

Further, as above, there exists \( \eta_1 \in (L,\eta), \eta_2 \in (\eta,M) \) such that

\[ \pi(L) - \pi(\eta) = \bar{F}(\eta_1)(\eta - L), \]

(3.13)
\[ \pi(\eta) - \pi(M) = \bar{F}(\eta_2)(M - \eta). \]

(3.14)

Inserted into (3.12) and using that \( \bar{F}(\eta_1) > \bar{F}(\eta_2) \), one obtains the inequality

\[
\begin{align*}
\eta \cdot (\eta - L) &+ \frac{1}{2} \left( \bar{F}(\eta_1) - \bar{F}(\eta_2) \right) \left[ \bar{F}(M) - \bar{F}(\eta_2) \right] \left[ \bar{F}(M) - \bar{F}(\xi) \right] \\
&> (M - L)^2 \left[ \bar{F}(\eta_2) \left( \bar{F}(M) - \bar{F}(\xi) \right) \right] + \left[ \bar{F}(\xi) - \bar{F}(\eta_2) \right] \left[ \bar{F}(M) - \bar{F}(\xi) \right] > 0,
\end{align*}
\]

(3.15)

which implies that \( r = \frac{1}{2} \) is the only solution of the equation (3.8) with \( M > L \). Since \( d = \frac{1}{2}(L + M) \) it follows from (C1)-(C3) that \( L \) is the unique solution of the equation \( \pi(L) + \pi(2d - L) = 2c \), where \( d = P - P_r \geq \mu \), \( M = 2d - L \). ∎

4. Pricing perfectly hedged contracts.

We follow the pricing methodology introduced in Hürlimann(1994), which has been extended and generalized in Hürlimann(1998).

An insurance contract is a pair \( \{X, P\} \) consisting of an insurance risk \( X \) and a risk premium \( P = H[X], \) where \( H[\cdot] \) is some premium calculation principle. An experience rated contract is a triple \( \{X, P, D\} \) consisting of an insurance contract \( \{X, P\}, \) which besides claims payment \( X \) offers a bonus or dividend \( D = D[X] \geq 0, \) paid out in case the financial gain \( P - X \) is positive. In particular, a perfectly hedged contract is a special experience rated contract.

Definition 4.1. The experience rating premium \( P \) of an experience rated contract \( \{X, P, D\} \) is called \( H\)-compatible if \( P = H[X]. \)

The liability of an experience rated contract is \( X + D. \) Suppose its financial valuation, also called risk premium, equals \( P = P[X + D], \) and is thus a functional depending on \( X \) and \( D \) through their sum. Compared to an ordinary insurance contract (without bonus or dividend payment) with financial loss \( X - P, \) the financial loss \( X + D - P \) of an experience rated contract may be much more important. To improve his level of security, suppose the insurer concludes a risk-exchange treaty, denoted REX, with some reinsurer. The REX consists of a pair \( \{Y, Z\} \) such that \( Y + Z = X, \) where \( Y = Y[X] \) is the retained amount of the cedant and \( Z = Z[X] \) is the random payment from the reinsurer to the cedant. Suppose the insurer chooses a REX from a feasible set \( S(X) \) of possible REX's. Feasible sets often considered are \( S(X) = \text{POREX}(X), \) the set of feasible Pareto-optimal REX's, or \( S(X) = \text{Com}(X) \) the set of feasible reinsurance contracts described by the class of comonotonic random variables defined in Section 2. Three main questions are:

(Q1) How does the insurer choose \( \{Y, Z\} \in S(X) \) and \( D? \)

(Q2) What is an adequate price \( P \) for an experience rated contract ?

(Q3) Which properties must the premium principle \( H[\cdot] \) satisfy ?
Having concluded a REX, the required premium \( P = P[X+D] \) of an experience rated contract is the sum of the net retained premium \( P_c = P_c[Y+D] \) of the cedant plus the price \( H[Z] \) paid to the reinsurer for the REX treaty, hence \( P = P_c + H[Z] \). To answer question (Q1), suppose the insurer applies a minimum \( p \)-norm financial loss principle as decision criterion. Thus, one has to minimize the \( p \)-norm of the difference between assets and liabilities of the insurer, that is one considers for some \( p > 1 \) the optimization problem

\[
R(p) = \left\| P_c - Y - D \right\|^p_p = E\left[\left\| P_c - Y - D \right\|^p\right] = \min.
\]

over all \((Y,Z) \in S(X)\) and all \( D \) from some set of dividend formulas. Questions (Q2) and (Q3) are related. Clearly the premium functional \( P = P[Y,Z,D] = P_c[Y+D] + H[Z] \) and \( H[] \) must satisfy some desirable properties. In the following, only three plausible properties will be relevant:

(P1) The no unjustified loading property: If \( X = c \) is a constant risk, then \( H[c] = c \).

(P2) The \( S(X) \)-additive property: If \( (Y,Z) \in S(X) \) then \( H[Y] + H[Z] = H[X] \).

(P3) The fair property for the retained business: The net retained premium of the insurer is a fair premium in the sense that \( P_c = E[Y+D] \).

The property (P2) says that no arbitrage profit can be made from concluding a REX treaty. In the important special case \( S(X) = \text{Com}(X) \), (P2) is the comonotonic additive property. It is in particular satisfied by the class of quantile premium principles, which include the absolute deviation principle and the Gini principle (e.g. Denneberg (1985/90), and Wang (1996)). Note that the comonotonic property is also essential in Chateauneuf et al. (1996). Other premium principles satisfying (P2) include the distribution-free principle derived in Hürlimann (1994), Theorem 5.1, and the \( p \)-norm principle introduced in Hürlimann (1998). The fair property (P3) is satisfied by the "fair premiums" required to cover the perfectly hedged experience rated contracts considered in Hürlimann (1994), Section 4, which correspond to the solutions of the minimization problem \( R(p = 2)_{\min} = 0 \) in (4.1) with \( S(X) = \text{Com}(X) \). This property implies that the insurer does not take any risk in this situation. In general, under (P3), the premium of an experience rated contract \( \{X,P,D\} \) equals \( P = E[Y+D] + H[Z] \) and may be decomposed in three components as follows:

\[
P = E[X] + E[D] + (H[Z] - E[Z]).
\]

If one requires additionally \( H \)-compatible premiums, then the following general characterization holds.

**Theorem 4.1.** Let \( \{X,P,D\} \) be an experience rated contract, which offers a claims dependent dividend under the help of a REX treaty \( (Y,Z) \in S(X) \), with premium
\[ P = E[Y + D] + H[Z] \] (fair property (P3)). Let \( H[\cdot] \) be a premium principle, which satisfies the properties (P1) and (P2). Then the experience rated premium is \( H \)-compatible, that is

\[ P = H[X] = H[Y] + H[Z], \quad \text{for all } (Y, Z) \in S(X), \quad (4.3) \]

if and only if \( H[\cdot] \) acts as follows on the retained business:

\[ H[Y] = E[Y + D]. \quad (4.4) \]

**Proof.** This is an immediate consequence of the formulas (4.2) and (4.3). ◊

**Definition 4.2.** An experience rated contract with \( H \)-compatible premium as in Theorem 4.1 will be called a \( H \)-fair experience rated contract.

To justify the validity of the pricing methodology proposed in Theorem 4.1 for general \( p \)-integrable insurance risks, e.g. Pareto claims with infinite variance (index \( 1 < p < 2 \)), it suffices to satisfy the fair property (P3) for some non-trivial class of experience rated contracts, which solve the optimization problem (4.1), for example such that \( R(p) = 0 \). In generalization of the case \( p = 2 \), this remains true for the class \( S_d(X) \) of perfectly hedged experience rated contracts with maximum deductible \( d \) in Section 2. Indeed, the premium of such a contract is given by

\[ P = P[X + D] = P[d + Z] = d + H[Z] \] (under the assumption of the very plausible translation-invariant property). It follows that

\[ P_c = P - H[Z] = d = Y + D = E[Y + D], \] which shows the required property (P3).

Consider now the pricing of an experience rated contract \( \{X, P, D\} \) subject to a feasible risk-exchange \( (Y, Z) \in S(X) \), with \( S(X) = POREX(X) \) or \( S(X) = Com(X) \). As motivated and justified in Hürlimann (1998), the following \( p \)-norm principle can be used to model market premiums in a competitive equilibrium environment. Let \( 1 < p < \infty \) and assume \( X \) belongs to the space \( L^p \) of all \( p \)-integrable random variables \( X \) with finite mean \( E[X] \) and \( p \)-th moment \( E[X^p] < \infty \). For a positive real number \( \lambda \), let \( x^{(\lambda)} = |x|^{\lambda} \cdot \text{sgn}(x) \) be the signed power function. If \( X, Y \in L^p \) one defines the \( p \)-product by

\[ \langle X, Y \rangle_p = E[(X - E[X]) \cdot (Y - E[Y])^{(p-1)}], \quad (4.5) \]

which induces the \( p \)-norm

\[ \|X\|_p^p = \langle X, X \rangle_p = E\|X - E[X]\|^p. \quad (4.6) \]

Then the market prices associated to \( X, Y, Z \) are given by

\[ H[X] = E[X] + \gamma^{(p-1)} \cdot \|X\|_p^p, \quad \gamma \in R, \]

\[ H[Y] = E[Y] + \frac{\langle Y, X \rangle_p}{\|X\|_p^p} \cdot (H[X] - E[X]), \quad (4.7) \]
\[ H[Z] = E[Z] + \frac{\langle Z, X \rangle_p}{\|X\|_p^p} (H[X] - E[X]). \]

Now, if the assumptions of Theorem 4.1 are fulfilled, then the experience rated premium will be H-compatible if and only if \( H[Y] = E[Y + D] \). Inserting into (4.7), the unknown constant \( \gamma \) can be eliminated, and one obtains the market premium

\[ P = H[X] = E[X] + \frac{\|X\|_p^p}{\langle X, Y \rangle_p} E[D], \quad (4.8) \]

and the corresponding reinsurance premium

\[ P_R = H[Z] = E[Z] + \frac{\langle Z, X \rangle_p}{\langle Y, X \rangle_p} E[D]. \quad (4.9) \]

These formulas extend the relations (4.16), (4.17) of Hürlimann(1994) in two directions. Firstly, they are valid for arbitrary \( p \)-integrable random variables, allowing models with infinite variance. Secondly, they are valid for arbitrary Pareto-optimal risk-exchanges, and not just for perfectly hedged experience rated contracts as in our first study. In particular, even in the classical case \( p = 2 \), a useful generalization has been obtained.

In our illustration to life insurance in Section 6, the variance of the risk is finite, thus the special case \( p = 2 \) applies with

\[ P = E[X] + \frac{\text{Var}[X]}{\text{Cov}[X, Y]} E[D], \]

\[ P_R = E[Z] + \frac{\text{Cov}[X, Z]}{\text{Cov}[X, Y]} E[D]. \quad (4.10) \]

5. **Reinsurance and economic capital.**

As a first step, the question of whether splitting a risk or not must be answered. In general, an insurance contract \( \{X, P\} \) is endowed with an initial capital \( u \), which during the insurance period is changing to the random value \( U \) according to the insurance risk process equation

\[ U = u + P - X. \quad (5.1) \]

An insurance company is said to be solvable provided \( U \) remains non-negative with a high probability \( \alpha \), that is

\[ \Pr(U \geq 0) \leq \alpha. \quad (5.2) \]

By fixed initial capital and confidence level \( \alpha \), the minimum risk premium satisfying (5.2) is called solvability premium. It is denoted and given by

\[ P_z = P_z(u, \alpha) = \text{VaR}_u[X] - u. \quad (5.3) \]
where \( \text{VaR}_\alpha[X] := \inf \{ x \mid F_X(x) \geq \alpha \} \) is the value-at-risk or \( \alpha \)-quantile of \( X \), and \( F_X(x) = \Pr(X \leq x) \) is the distribution function of the risk. In case the risk premium is strictly less than the solvability premium, that is if the inequality

\[
u + P < \text{VaR}_\alpha[X]
\]

is satisfied, then there is insolvency risk. In this situation, either \( u \) and/or \( P \) have to be increased or a risk-exchange/reinsurance must be considered.

To cover the insolvency risk, an insurance company holds economic capital (EC) as measured by value-at-risk (VaR) or, to be “coherent”, by conditional value-at-risk (CVaR) (Rockafellar and Uryasev(2001), Hürlimann(2001)).

For simplicity, let us define EC as follows. The future random loss of the portfolio can be decomposed as follows:

\[
X - P = (\mu - P) + (X - \mu).
\]

The first component, which is the negative of the insurance margin, represents the future expected insurance gain and belongs to the stakeholders of the insurance company. To protect this expected gain, one requires some economic capital to cover the insurance loss \( X - \mu \) (signed deviation from the mean aggregate claims). Using VaR the future value of this economic capital is equal to

\[
\text{EC}_\alpha[X] = \text{VaR}_\alpha[X - \mu] = \text{VaR}_\alpha[X] - \mu,
\]

where \( \alpha \) is some prescribed confidence level.

After a reinsurance, the economic capital associated to the retained risk \( X_C \) will be

\[
\text{EC}_\alpha[X_C] = \text{VaR}_\alpha[X_C - \mu_C] = \text{VaR}_\alpha[X_C] - \mu_C,
\]

where \( \mu_C = E[X_C] \) is the expected value of the retained risk.

In general, holding economic capital is expensive because shareholders expect a higher return on risky investments than that expected from a risk-free investment. Assuming solvability, a main goal of an insurance company is to achieve (with or without reinsurance) a possibly small cost of capital (CoC). To define the latter quantity, suppose that \( i \) is the one-year interest rate to be paid on the required economic capital (net of the available initial capital). In case the latter quantity is negative, the cost of capital is a gain (in fact, part of the initial capital may be reimbursed to the stakeholders of the insurance company). Then one has

\[
\text{CoC}_\alpha[X] = i \cdot (\text{EC}_\alpha[X] - u).
\]

After a reinsurance with payoff \( Z \), the cost of capital, which includes the transfer of expected profit, is equal to

\[
\text{CoC}_\alpha[X_C] = P_R - \mu_R + i \cdot (\text{EC}_\alpha[X_C] - u),
\]
where $\mu_r = \mu - \mu_c$ is the expected payment from the reinsurance contract. Clearly, a reinsurance payoff $Z^1$ with premium $P^1_r$ is preferred to $Z^2$ with premium $P^2_r$, and both are preferred to the solution without reinsurance provided

$$CoC_a[X^1_c] \leq CoC_a[X^2_c] \leq CoC_a[X].$$

(5.10)

In the special case of a perfectly hedged contract $(Y, Z) \in S_c(X)$ with maximum deductible $d$, the cedant retains the riskless constant amount $X_c = X + D - Z = d$, and the cedant’s cost of capital reduces to the quantity

$$CoC_a[X_c] = P_r - \mu_r - i \cdot u.$$  (5.11)

6. Illustration from life insurance.

The results of the present paper are illustrated at a risk $X$ from life insurance corresponding to a portfolio of $N$ identical life insurance policies with sums at risk death $T = \text{'000'000}$ for $N = 1000, 2000$, where the probability of death is assumed to be $q = 0.005$. To decide the question whether the insurer should enter a reinsurance or not, we compare the cost of capital before and after reinsurance as defined in Section 5. Our attention is restricted to the two-layer’s stop-loss contract $Z = r(X - L)_+ + (1 - r)(X - M)_+$, whose optimal parameters are determined by Theorem 3.1. The market risk premium and the reinsurance premium of this perfectly hedged contract are determined by the formulas (4.10). To implement the latter, expressions for the covariance between $X$ and $Y = X - Z$ are required. A straightforward calculation yields the formulas

$$Cov[X, Y] = \sigma^2 - Var[Z] - Cov[Y, Z].$$

(6.1)

$$Var[Z] = r^2 \sigma^2(L) + (1 - r)^2 \sigma^2(M) + 2r(1 - r)\pi(M)\{M - L - [\pi(L) - \pi(M)]\},$$

(6.2)

$$Cov[Y, Z] = E[Z](d - \mu + E[Z]) + r(1 - r)[\pi_2(L) - \pi_2(M) - (M - L)[\pi(L) + \pi(M)]].$$

(6.3)

As market premiums are often quoted in terms of the expected value principle, we consider the loading factors $\theta, \theta_r$ such that

$$P = (1 + \theta) \cdot \mu, \quad P_r = (1 + \theta_r) \cdot E[Z].$$

(6.4)

Using Theorem 3.1 and comparing (6.4) with (4.10) one sees that the loading factors are necessarily determined by the functions

$$\theta = \left(\frac{\sigma^2}{\sigma^2 - Var[Z] - Cov[Y, Z]}\right) \left(\frac{d - \mu + c}{\mu}\right),$$

(6.5)

$$\theta_r = \frac{(1 + \theta)\mu - d - c}{c},$$

(6.6)

where $Var[Z], Cov[Y, Z]$ are given by (6.2), (6.3) with $r = \frac{1}{2}$, and

$$d = \frac{1}{2}(L + M), \quad c = E[Z] = \frac{1}{2}(\pi(L) + \pi(M)).$$

(6.7)
The limiting case $M \to L$ converges to the stop-loss contract $Z = (X - d)_+$ with deductible $d = L$. In practice, often $P$ and $\theta$ are known, but $\theta_r$ is unknown. In this situation, one proceeds as follows. Inserting the first equation from (4.10) into the second, one obtains

$$\theta_r c = \left( \frac{\sigma^2 - \text{Cov}[X,Y]}{\text{Cov}[X,Y]} \right) \cdot (d - \mu + c) = \left( \frac{\sigma^2 - \text{Cov}[X,Y]}{\sigma^2} \right) \cdot \frac{\theta \mu}{\sigma^2}$$

$$= (\text{Var}[Z] + \text{Cov}[Y,Z]) \cdot \frac{\theta \mu}{\sigma^2}. \quad (6.8)$$

Using (6.6) it follows that the optimal parameters $L, M$ solve the implicit equation

$$(1 + \theta) \mu - d - c - (\text{Var}[Z] + \text{Cov}[Y,Z]) \cdot \frac{\theta \mu}{\sigma^2} = 0, \quad (6.9)$$

with $c$ and $d$ as in (6.7), and the unknown loading factor $\theta_r$ of the reinsurer is obtained from (6.6). Applying (5.11) the cost of capital after reinsurance is given by

$$\text{CoC}_a[X_c] = P - \frac{1}{2}(L + M + \pi(L) + \pi(M)) - i \cdot u, \quad (6.10)$$

where $L, M$ solve (6.9). The reduction in cost of capital is given by

$$\Delta \text{CoC}_a[X, X_c] = \text{CoC}_a[X] - \text{CoC}_a[X_c]$$

$$= i \cdot E_a[X] + \frac{1}{2}(L + M + \pi(L) + \pi(M)) - P. \quad (6.11)$$

Numerical calculations show that by fixed risk premium $P$ (or fixed $\theta$) the reduction is maximal for the limiting stop-loss contract with maximum deductible $d = L = M$. In this case, raising the risk premium automatically increases the random bonus payment $D = (d - X)_+$. The heavy loading factor of the reinsurer increases faster than the loading factor of the risk premium in accord with the explanation in Amsler (1986). Tables 6.1 and 6.2 summarize some calculations.

**Table 6.1**: CoC for the limiting optimal stop-loss contract, with the values

$\alpha = 99.8\%, u = 2000\,000, i = 8\%$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\theta$</th>
<th>$\theta_r$</th>
<th>$L = M$</th>
<th>$\text{CoC}_a[X]$</th>
<th>$\Delta \text{CoC}_a[X, X_c]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.5</td>
<td>1.959</td>
<td>5'513'287</td>
<td>546'615</td>
<td>-608'732</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>2.845</td>
<td>6'517'491</td>
<td></td>
<td>-390'367</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>3.837</td>
<td>7'379'882</td>
<td></td>
<td>-181'948</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>4.927</td>
<td>8'147'667</td>
<td></td>
<td>-1'924</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>6.111</td>
<td>8'849'082</td>
<td></td>
<td>147'228</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>7.387</td>
<td>9'502'293</td>
<td></td>
<td>268'252</td>
</tr>
<tr>
<td>2000</td>
<td>0.5</td>
<td>3.837</td>
<td>13'821'701</td>
<td>760'087</td>
<td>-14'619</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>5.339</td>
<td>15'241'846</td>
<td></td>
<td>281'539</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>7.017</td>
<td>16'512'657</td>
<td></td>
<td>493'530</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>8.871</td>
<td>17'688'984</td>
<td></td>
<td>640'580</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>10.901</td>
<td>18'803'744</td>
<td></td>
<td>740'322</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>13.109</td>
<td>19'877'883</td>
<td></td>
<td>806'625</td>
</tr>
</tbody>
</table>
In some cases, for example $N = 2000, \theta = 1$ in Table 6.1, it is possible to reduce the initial capital because the cost of capital after reinsurance is negative. Even when a stop-loss contract is not available, the tow-layers stop-loss substitute provides enough flexible product design. Table 6.2 exemplifies this situation choosing the limits $L$ and $M$, which automatically determine the loading factors according to (6.5) and (6.6).

**Table 6.2**: CoC for the two-layers optimal stop-loss contract, with the values $\alpha = 99.8\%, u = 0, i = 8\%, N = 2000, CoC_a[X] = 920'087$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\theta_r$</th>
<th>$L$ (in Mio.)</th>
<th>$M$ (in Mio.)</th>
<th>$\Delta CoC_a[X, X_c]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.512</td>
<td>4.005</td>
<td>14</td>
<td>14</td>
<td>25'289</td>
</tr>
<tr>
<td>0.548</td>
<td>4.467</td>
<td>14</td>
<td>15</td>
<td>118'438</td>
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<tr>
<td>0.589</td>
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<td>14</td>
<td>16</td>
<td>178'827</td>
</tr>
<tr>
<td>0.582</td>
<td>5.056</td>
<td>15</td>
<td>15</td>
<td>235'088</td>
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<tr>
<td>0.621</td>
<td>5.599</td>
<td>15</td>
<td>16</td>
<td>316'210</td>
</tr>
<tr>
<td>0.664</td>
<td>6.072</td>
<td>15</td>
<td>17</td>
<td>367'700</td>
</tr>
<tr>
<td>0.659</td>
<td>6.300</td>
<td>16</td>
<td>16</td>
<td>414'684</td>
</tr>
<tr>
<td>0.700</td>
<td>6.929</td>
<td>16</td>
<td>17</td>
<td>481'213</td>
</tr>
<tr>
<td>0.745</td>
<td>7.461</td>
<td>16</td>
<td>18</td>
<td>522'489</td>
</tr>
<tr>
<td>0.744</td>
<td>7.750</td>
<td>17</td>
<td>17</td>
<td>560'060</td>
</tr>
<tr>
<td>0.784</td>
<td>8.466</td>
<td>17</td>
<td>18</td>
<td>611'826</td>
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<tr>
<td>0.831</td>
<td>9.057</td>
<td>17</td>
<td>19</td>
<td>643'214</td>
</tr>
</tbody>
</table>

**References.**


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