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Economic risk capital allocation from top down

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Suppose an insurance risk business consists of \( n \) different planning units \( U_i, \ i = 1, \ldots, n \), which are each subdivided in \( m_i \) product categories \( C_{ij}, \ j = 1, \ldots, m_i \). The insurance risks during some insurance period, which are associated to all these different business lines, are measured by the risk process random variables

\[
X_i = S_i - P_i, \quad i = 1, \ldots, n, \\
X_{ij} = S_{ij} - P_{ij}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n,
\]

where \( P_i, P_{ij} \), respectively \( S_i, S_{ij} \), denote the risk premiums, respectively the aggregate claims random variables, which are associated to \( U_i, C_{ij} \). Consider the following main risk characteristics of the aggregate claims distributions:

\[
\mu_i = \mathbb{E}[S_i] : \text{the mean aggregate claims of } U_i, \quad i = 1, \ldots, n \\
\mu_{ij} = \mathbb{E}[S_{ij}] : \text{the mean aggregate claims of } C_{ij}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n \\
\kappa_i = \frac{\sqrt{\text{Var}[S_i]}}{\mathbb{E}[S_i]} : \text{the coefficient of variation of } U_i, \quad i = 1, \ldots, n \\
\kappa_{ij} = \frac{\sqrt{\text{Var}[S_{ij}]}}{\mathbb{E}[S_{ij}]} : \text{the coefficient of variation of } C_{ij}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n.
\]

To protect the planning units and the product categories against random fluctuations in the risk processes, one is interested in the evaluation of economic risk capital for the planning units and a corresponding adequate allocation of economic risk capital to the product categories. The risk processes, which describe the future random losses of the planning units and product categories can be decomposed as follows:

\[
X_i = (S_i - \mu_i) + (\mu_i - P_i), \quad i = 1, \ldots, n, \\
X_{ij} = (S_{ij} - \mu_{ij}) + (\mu_{ij} - P_{ij}), \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n.
\]

The second component of these expressions, which is the negative of the insurance margin, represents the future expected insurance gain and belongs to the stakeholders of the insurance company. To protect the expected gains, one requires economic risk capital to cover the insurance losses \( L_i = S_i - \mu_i, L_{ij} = S_{ij} - \mu_{ij} \). Using the simple and ubiquitous value-at-risk measure to define economic risk capital, the future values of the economic risk capital (ERC) are defined by the quantities

\[
\text{ERC}_i = \text{VaR}_\alpha[L_i], \quad i = 1, \ldots, n, \\
\text{ERC}_{ij} = \text{VaR}_\alpha[L_{ij}], \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n,
\]

where \( \alpha \) is a sufficiently high confidence level (e.g. \( \alpha = 0.99 \)), and value-at-risk of a random variable \( X \) is defined to be the \( \alpha \)-quantile \( Q_X(\alpha) \) obtained from the distribution function \( F_X(x) \) through the relation

\[
\text{VaR}_\alpha[X] = Q_X(\alpha) = \inf\{x : F_X(x) \geq \alpha\}.
\]

Recall that the VaR quantity represents the maximum possible loss, which is not exceeded with the probability \( \alpha \).
For practical purposes, we will assume that the distribution functions of $S_i, S_{ij}$ can be well approximated by gamma distributions such that (Appendix B, Theorem 2)

$$S_i \sim \Gamma \left( \frac{1}{k_i^2}, \frac{1}{k_i^2 \mu_i} \right), \quad i = 1, \ldots, n,$$

$$S_{ij} \sim \Gamma \left( \frac{1}{k_{ij}^2}, \frac{1}{k_{ij}^2 \mu_{ij}} \right), \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n.$$

It is a straightforward exercise to derive the following ERC-formulas:

$$\text{ERC}_i = \left( \Gamma_{\alpha}^{-1} \left( \frac{1}{k_i^2} k_i^2 - 1 \right) \right) \cdot \mu_i, \quad i = 1, \ldots, n,$$

$$\text{ERC}_{ij} = \left( \Gamma_{\alpha}^{-1} \left( \frac{1}{k_{ij}^2} k_{ij}^2 - 1 \right) \right) \cdot \mu_{ij}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n,$$

where $\Gamma_{\alpha}^{-1}(\beta)$ denotes the $\alpha$-quantile of the “standardized” gamma distribution $\Gamma(\beta, 1)$. Appendix C, Theorem 3, shows that for compound Poisson aggregate claims models with gamma claim size densities (justified in Appendix B), these ERC approximations yield ERC upper bounds, at least for sufficiently high confidence levels and sufficiently large expected number of claims. The diversification effect between the insurance risks of the products in each planning unit, defined as difference between the sum of the ERC measures of the standalone product risks and the ERC measure of the planning unit risk, is denoted and given by

$$D_i = \sum_{j=1}^{m_i} \text{ERC}_{ij} - \text{ERC}_i, \quad i = 1, \ldots, n.$$

In practice, the statistical analysis of the risk processes associated to the different product categories is usually difficult and cumbersome, and a reliable estimation of the parameters $\mu_{ij}, k_{ij}$ is seldom available. Therefore, it is very desirable to develop methods, which produce useful “top down” estimators such that to each triple $(P_i, \mu_i, k_i)$ one associates triples $(P_{ij}, \mu_{ij}, k_{ij})$. The advantage of such a procedure is a simple and direct calculation of the economic risk capitals $\text{ERC}_{ij}$ as well as the corresponding diversification effects $D_i$. In the Appendix A, Theorem 1, it is shown that a simple homogeneity assumption on the claim sizes of the product categories leads to such a practical method. Following the rules of Theorem 1, one obtains under this assumption the formulas

$$\mu_{ij} = \left( \frac{\mu_i}{P_i} \right) \cdot P_{ij}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n,$$

$$k_{ij} = \left( \frac{P_i}{P_{ij}} \right) k_i, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, n.$$

The implementation of this method requires only the knowledge of the risk premiums $P_{ij}$ for each product category as well as statistical estimation of the parameters $\mu_i, k_i$ for each planning unit.

Appendix A: Risk theoretical justification of top down allocation

Consider a planning unit divided in $m$ product categories. The insurance risk is described by the following quantities:

- $P$ : risk premium of a planning unit
- $S$ : aggregate claims random variable of a planning unit
- $\mu = E[S]$ : mean aggregate claims of a planning unit
- $k = \sqrt{\text{Var}[S]} / E[S]$ : the coefficient of variation of a planning unit
Applying the collective model of risk theory, one approximates the aggregate claims random variables \( S_i \) by compound Poisson random variables

\[
S_i = \sum_{j=1}^{N_i} Y_{i,j}, \quad i = 1, \ldots, m, \tag{1}
\]

where \( N_i \) represents the number of claims and \( Y_{i,j} \) represents the claim size given a claim has occurred. The model supposes that \( N_i \) is Poisson(\( \lambda_i \)) distributed, the \( Y_{i,j} \)'s are independent and identically distributed and they are independent from \( N_i \) (e.g. Beard et al. (1984), Section 3). Denote the identical claim sizes by \( Y_i = Y_{i,j}, \quad j = 1, \ldots, N_i \). Furthermore, assume that the aggregate claims of the product categories are independent random variables. Then the aggregate claims random variable \( S \) of the planning unit is again compound Poisson such that

\[
S = \sum_{i=1}^{N} Z_i, \tag{2}
\]

where \( N \) is Poisson(\( \lambda \)) distributed, with \( \lambda = \sum_{i=1}^{m} \lambda_i \), the \( Z_i \)'s are independent and identically distributed, with \( Z_i = \sum_{j=1}^{N_i} \frac{\lambda_i}{\lambda} \cdot Y_i \), and the \( Z_i \)'s are independent from \( N \). The identical claim sizes are denoted by \( Z = Z_i, \quad i = 1, \ldots, N \). Under a simple homogeneity assumption, the main risk parameters of the product categories relate to those of the planning unit as follows.

\[ \text{Theorem 1} \]

In the above collective model of risk theory, suppose that the claim sizes of the product categories are identically distributed, that is \( Y_i = Y \) for all \( i = 1, \ldots, m \). Assume the risk premiums are calculated according to the expected value principle such that \( P = (1 + \theta) \cdot \mu, \quad P_i = (1 + \theta) \cdot \mu_i, \quad i = 1, \ldots, m, \) with \( \theta \) the loading factor. Then the means and coefficients of variation \((\mu_i, k_i), \quad i = 1, \ldots, m\), relate to \((\mu, k)\) according to the following rules:

\[ \text{(R1)} \quad \frac{\mu_i}{P_i} = \frac{\mu}{P}, \quad i = 1, \ldots, m \]

(invariance of the expected aggregate claims per unit of risk premium)

\[ \text{(R2)} \quad k_i = \sqrt{\frac{P}{P_i}} \cdot k, \quad i = 1, \ldots, m \]

(coefficients of variation inverse proportional to the square-root risk premiums)

\[ \text{Proof:} \]

Define the following quantities, which are related to the first and second moments of the claim sizes:

\[
\nu_i = \text{E}[Y_i], \quad i = 1, \ldots, m, \quad \nu = \text{E}[Z],
\]

\[
\nu_i^2 = \frac{\text{E}[Y_i^2]}{\nu_i^2}, \quad i = 1, \ldots, m, \quad \nu^2 = \frac{\text{E}[Z^2]}{\nu^2}.
\]
Under the compound Poisson assumption, one has the relations
\[ \mu_i = E[N_i] \cdot E[Y_i] = \lambda_i v_i, \quad i = 1, \ldots, m, \quad \mu = E[N] \cdot E[Z] = \lambda v. \]
\[ (k_i \mu_i)^2 = Var[S_i] = E[N_i] \cdot E[Y_i^2] = \lambda_i c_i^2 v_i^2, \quad i = 1, \ldots, m. \]
\[ (k \mu)^2 = Var[S] = E[N] \cdot E[Y] = \lambda c^2 v^2. \]
Through comparison one gets the relations
\[ k_i^2 = \frac{c_i^2}{\lambda_i}, \quad i = 1, \ldots, m. \]
\[ k^2 = \frac{c^2}{\lambda}. \]
Under the homogeneity assumption \( Y_i = Y, \ i = 1, \ldots, m, \) one has \( Z = \sum_{i=1}^{m} \lambda \cdot Y = Y, \) hence \( v_i = v, \ c_i = c, \ i = 1, \ldots, m. \) It follows that \( \lambda k_i^2 = c^2 = c_i^2 = \lambda_i k_i^2, \ i = 1, \ldots, m, \) which is equivalent to the rule (R2). Similarly, from the defining expressions for the risk premiums one gets the relations
\[ \frac{\mu_i}{p_i} = (1 + \theta)^{-1} = \frac{\mu}{p}, \quad i = 1, \ldots, m, \]
which shows the rule (R1). \( \square \)

Appendix B: Gamma approximation of the aggregate claims distribution

Suppose the aggregate claims random variable of an insurance risk portfolio can be represented by a compound Poisson random variable
\[ S = \sum_{i=1}^{N_i} Y_i, \]
where \( N \) is Poisson(\( \lambda \)) distributed, the \( Y_i's \) are independent and identically distributed non-negative random variables, which are stochastically independent from \( N. \) Denote the identical random variables by \( Y = Y_i, \ i = 1, \ldots, N. \) From a practical viewpoint, it has been stated for a long time that a gamma approximation of the claim size is appropriate for modeling the insurance risk process in life insurance (e.g. OECD (1971), Strickler (1982), Drude (1988), p. 183, H"urlimann (1988)). Theoretically, this claim size model arises as unique solution of a characterization problem for scale compound parametric families of distributions with the mean as scale parameter (Proposition 3.2 in H"urlimann (1998)). Let us follow this approach. A gamma claim size has density
\[ f_Y(x) = g(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0, \]
with the parameters
\[ \alpha = \frac{m_2}{m_2 - m_1}, \quad \beta = \frac{m_1}{m_2}, \quad m_i = E[Y_i], \quad i = 1, 2. \]
The distribution function of the claim size is given by the incomplete gamma function
\[ F_Y(x) = G(\beta x; \alpha) = \frac{1}{\Gamma(\alpha)} \int_0^{\beta x} t^{\alpha-1} e^{-t} dt. \]
Using Laplace transforms one sees that the j-th convolution of the claim size density equals

\[ f_{Y_j}^x(x) = g(x; ja, \beta), \quad j = 1, 2, \ldots. \]

It follows that the aggregate claims distribution function has the representation

\[ F_{S}(x; \lambda, a, \beta) = e^{-\lambda} + \sum_{j=1}^{\infty} e^{-\lambda} \frac{\lambda^j j!}{j!} \cdot G(\beta x; ja). \]

The following result justifies the use of the gamma approximation to the aggregate claims distribution for sufficiently large portfolios.

**Theorem 2**

Under the above assumptions, the compound Poisson gamma distribution of the aggregate claims converges to a gamma distribution as the expected number of claims tends to infinity. More precisely, with the parameter function of one variable \( a(\lambda) = (k^2 - \lambda^{-1})^{-1} \) such that \( \alpha = a(\lambda) = \lambda^{-1} \cdot a(\lambda) \), \( \beta = \beta(\lambda) = \mu^{-1} \cdot a(\lambda) \), the limiting distribution is given by

\[ \lim_{\lambda \to \infty} F_{S}(x; a, a(\lambda), \beta(\lambda)) = G(bx; a). \]

**Proof:** A simple calculation shows the validity of the limit

\[ \lim_{\lambda \to \infty} \lambda f_Y(x; a, \beta) = \lim_{\lambda \to \infty} \left\{ \frac{a(\lambda) f_Y(x; \beta(\lambda))}{a(\lambda) + 1} \cdot e^{-\beta(x)} \right\} = \frac{a}{x} \cdot e^{-bx}. \]

Using Dufresne et al. (1991), Section 2, this asymptotic result identifies \( \lim_{\lambda \to \infty} F_{S}(x) \) with a gamma process whose distribution function equals \( G(bx; a) \).

**Appendix C: Value-at-risk upper bound**

As in Appendix B, Theorem 2, suppose the aggregate claims random variable \( S \) has the compound Poisson gamma distribution

\[ F_{S}(x; a, \lambda, \beta) = e^{-\lambda} + \sum_{j=1}^{\infty} e^{-\lambda} \frac{\lambda^j j!}{j!} \cdot G(\beta(\lambda)x; ja(\lambda)). \]

which should be compared with the limiting gamma distribution \( G(bx; a) \). Our analysis applies standard results from ordering of risks theory (e.g. Kaas et al. (1994)).

**Definitions**

Let \( X, Y \) be random variables with distribution functions \( F_X(x) \), \( F_Y(x) \) and densities \( f_X(x) \), \( f_Y(x) \). Consider the following stochastic orders:

- **(SD)** \( X \) precedes \( Y \) in the stochastic dominance of first order, written \( X \preceq_{SD} Y \), if \( F_X(x) \geq F_Y(x) \) for all \( x \) in the common support of \( X \) and \( Y \).

- **(LR)** \( X \) precedes \( Y \) in the likelihood ratio order, written \( X \preceq_{LR} Y \), if the ratio of likelihoods \( f_X(x)/f_Y(x) \) is a decreasing function on the common support of \( X \) and \( Y \).

It is well-known that \( X \preceq_{LR} Y \Rightarrow X \preceq_{SD} Y \) (Kaas et al. (1994), Section V.1).
Firstly, we compare the gamma distributed random variables
\[ X_j \sim F(j \alpha(\lambda), \beta(\lambda)), \quad j = 1, 2, \ldots \]
\[ X \sim F(a, b) \]
where we suppose that \( \alpha = \alpha(\lambda), \beta = \beta(\lambda) > 0 \) (this holds for sufficiently large \( \lambda \)).

**Lemma**

The random variables \( X_j \) and \( X \) satisfy the following relationships:

**Case 1:** if \( j \leq \lambda - k^{-2} \) then \( X_j \leq_{LR} X \), hence \( X_j \leq_{SD} X \).

**Case 2:** if \( j > \lambda - k^{-2} \) there exists a constant \( c(j) \) such that the tails satisfy the relation \( F_{X_j}(x) \geq F_X(x) \) for all \( x \geq c(j) \).

**Proof:** Let \( f_j(x) = g(x; j \alpha, \beta) \) and \( f(x) = g(x; a, b) \) be the gamma densities of \( X_j \) and \( X \). The likelihood ratio \( q(x) = \frac{f_j(x)}{f(x)} \) has the derivative
\[ q'(x) = \frac{\Gamma(a) \beta^a}{\Gamma(j \alpha b^a)} x^{j \alpha - a - 1} e^{-(\beta - b)x} \cdot [(j \alpha - a) - (\beta - b)x]. \]

It is not difficult to see that
\[ \beta - b = \frac{1}{\mu} \left[ a(\lambda) - \frac{1}{k^2} \right] > 0, \quad j \alpha - a = \frac{j}{\lambda} a(\lambda) - \frac{1}{k^2} \leq 0 \quad \Leftrightarrow \quad j \leq \lambda - \frac{1}{k^2}. \]

Therefore, in Case 1 the derivative is negative, hence \( q(x) \) is decreasing, which shows that \( X_j \leq_{LR} X \).

In Case 2 the function \( q(x) \) has a maximum \( q(x_0) > 1 \) at \( x_0 = \frac{j \alpha - a}{\beta - b} \), and the difference \( \Delta(x) = f_j(x) - f(x) = f(x) \cdot [q(x) - 1] \) has two sign changes in the order \(-, +, -\). It follows that \( f_j(x) - f(x) \) has one sign change from \(-\) to \(+\) (e.g. Kaas et al. (1994), proof of Theorem III. 1.4).

Secondly, we compare the tails of the compound Poisson gamma distribution and its limiting gamma approximation.

**Theorem 3**

Under the assumptions of Theorem 2 one has the asymptotic inequality
\[ \lim_{x \to \infty} F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) \geq e^{-\lambda} + (1 - e^{-\lambda}) \cdot \lim_{x \to \infty} G(bx; a) \]
for all sufficiently high \( \lambda \) with \( \alpha(\lambda), \beta(\lambda) > 0 \).

**Proof:** Using the infinite series gamma representation of \( F_S \) and the Lemma, there exists for all integers \( N \geq 1 \) a constant \( c(N) \) such that
\[ F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) \geq e^{-\lambda} + \left( \sum_{j=1}^{N} \frac{e^{-\lambda \frac{j}{\lambda}}}{j!} \right) \cdot G(bx; a) \]
for all \( x \geq c(N) \). Taking limits as \( x \to \infty \) and \( N \to \infty \) shows the assertion. \( \square \)

The preceding result implies that for sufficiently high confidence levels \( \alpha \) and sufficiently large \( \lambda \) the value-at-risk of the aggregate claims random variable \( S \) is bounded above by \( \text{VaR}_\alpha[S] \leq \text{VaR}_\alpha[X] \), where \( X \) is the gamma approximation to \( S \). This inequality yields also an upper bound for ERC as defined in the text. For a numerical example, set \( \mu = 0.7, k = 0.2, \lambda = 100 \) and \( \alpha = 0.99 \). One obtains the upper bound \( \text{ERC}_\alpha[S] = \text{VaR}_\mu[S] - \mu = 0.361 \leq \text{ERC}_\alpha[X] = \text{VaR}_\mu[X] - \mu = 0.366 \). Since the parameters are subject to a considerable estimation uncertainty (especially for the coefficient of variation), the proposed simple gamma approximation can be recommended for practical purposes.
REFERENCES


Summary

Economic risk capital allocation from top down

To protect product categories in planning units of an insurance risk business against random fluctuations in the risk process, we propose a practical top down allocation of the economic risk capital (ERC). This method is justified using a simple homogeneity assumption on the claim sizes of the product categories. ERC is evaluated with a gamma approximation to the aggregate claims distribution. This approximation is justified for sufficiently large portfolios applying known properties of the gamma process. Moreover, for compound Poisson aggregate claims models with gamma claim size densities, it is shown that our ERC approximations yield ERC upper bounds, at least for sufficiently high confidence levels and sufficiently large expected number of claims.

Zusammenfassung

Vertikale Zuteilung des ökonomischen Risikokapitals