CHAPTER VIII. PORTFOLIO INSURANCE

1. Introduction.

Portfolio insurance, introduced by Leland in the night of September 11, 1976, is a simple financial instrument used to protect capital against future adverse falls. A collection of seminal papers, which study some of its main properties, is Luskin(1988). Combined with classical actuarial contingencies like mortality risk, portfolio insurance leads to unit-linked insurance contracts, which are analyzed in Chapter XII.

In the present Chapter, the focus on classical portfolio insurance reveals a new remarkable feature. It turns out that the economic risk capital as measured by value-at-risk or conditional value-at-risk remains constant provided the loss probability is sufficiently small. In practice, confidence levels $\alpha$ around 70%-90% often suffice to guarantee this stability property.

In Section 2, we derive a formula for the conditional value-at-risk of portfolio insurance, and show that it is constant for small loss probabilities. Two specific examples illustrate our results. In Section 3, we discuss portfolio insurance for an equity market index on the basis of empirical data material. The more general multivariate situation of a portfolio of risky assets is exemplified in Section 4 at the bivariate case.


Suppose that the random variable $S$ represents the market value of a portfolio of assets at some future date $T$. The goal of portfolio insurance consists to protect this future market value in such a way that the fixed value or limit $L$ is guaranteed. For this, an investor can either hold the assets and buy a put option with exercise price $L$ or hold cash at the risk-free continuous interest rate $\delta$ and buy a call option with exercise price $L$. The future values of these equivalent option strategies satisfy the identity

$$S + (L - S)_+ = L + (S - L)_+.$$  \hspace{1cm} (2.1)

Let $S_0$ denote the present value of the portfolio of assets, and let $P(L), C(L)$ be the put and call option prices with exercise price $L$, which are to be paid for these option strategies. Then the total cost $K(L)$ of portfolio insurance satisfies the put-call parity relation

$$K(L) = S_0 + P(L) = L \cdot e^{-\delta T} + C(L).$$  \hspace{1cm} (2.2)

The financial gain at time $T$ per unit of invested capital is described by the random return

$$R = \frac{L + (S - L)_+ - K(L)}{K(L)}.$$  \hspace{1cm} (2.3)

The potential investor decides upon investment by looking at the tradeoff between expected return and risk. Since the distribution of return is here asymmetrical, the usual variance as measure of risk cannot be recommended. As simple alternative we use the CVaR measure introduced in Section II.2.
\[ CVaR_{\alpha}[X] = Q_X(\alpha) + \frac{1}{\epsilon} \cdot \pi_X[Q_X(\alpha)] , \]  

(2.4)

where \( Q_X(\alpha) \) is the \( \alpha \)-quantile of \( X \), \( \pi_X(x) = E[(X - x)_+] \) is the stop-loss transform, and \( \epsilon = 1 - \alpha \) is interpreted as loss probability. Assuming the mean of \( X \) exists, we derive the following formula.

**Proposition 2.1.** The conditional value-at-risk associated to the negative return of portfolio insurance is determined by

\[ CVaR_{\alpha}[X] = \frac{1}{K(L)} \left\{ K(L) - L - (Q_S(\epsilon) - L)\pi + \frac{1}{\epsilon} \cdot (\pi_X[Q_X(\epsilon)] - \pi_X[L]) \right\} , \]

(2.5)

where \( \pi_X(x) = E[(x - X)_+] \) denotes the conjugate stop-loss transform.

**Proof.** The function \( I(E) \) of the event \( E \) denotes an indicator such that \( I(E) = 1 \) if \( E \) is true and \( I(E) = 0 \) otherwise. The evaluation of the distribution and stop-loss transform of \( X \) is done using the following separation into two steps:

\[ F_X(x) = \Pr\{X \leq x\} \cap \{S > L\} + \Pr\{X \leq x\} \cap \{S \leq L\} , \]

(2.6)

\[ \pi_X(x) = E[(X - x)_+] \cdot I\{S > L\} + E[(X - x)_+] \cdot I\{S \leq L\} . \]

(2.7)

To simplify notations, one sets \( \alpha = K(L)^{-1} \) and \( \beta(x) = (1 - x) \cdot K(L) \). Using (2.3) one sees that \( X \leq x \) if and only if \( \beta(x) \leq L + (S - L)_+ \). It follows without difficulty that

\[ \{X \leq x\} \cap \{S > L\} = \begin{cases} \{S > \beta(x)\}, & \beta(x) > L, \\ \{S > L\}, & \beta(x) \leq L, \end{cases} \]

(2.8)

\[ \{X \leq x\} \cap \{S \leq L\} = \begin{cases} \emptyset, & \beta(x) > L, \\ \{S \leq L\}, & \beta(x) \leq L. \end{cases} \]

(2.9)

Inserted in (2.6) one gets

\[ F_X(x) = \begin{cases} \bar{F}_S[\beta(x)], & \beta(x) > L, \\ 1, & \beta(x) \leq L, \end{cases} \]

(2.10)

from which one derives the \( \alpha \)-quantile expression

\[ Q_X(\alpha) = \begin{cases} \frac{K(L) - L}{K(L)}, & Q_S(\epsilon) \leq L, \\ \frac{K(L) - Q_S(\epsilon)}{K(L)}, & Q_S(\epsilon) > L. \end{cases} \]

(2.11)

Similarly, one has \( X > x \) if and only if \( \beta(x) \leq L + (S - L)_+ \), and one obtains that

\[ (X - x)_+ \cdot I\{S \leq L\} = \alpha \cdot (\beta(x) - L)_+ \cdot I\{S \leq L\} , \]

(2.12)

\[ (X - x)_+ \cdot I\{S > L\} = \alpha \cdot (\beta(x) - S)_+ \cdot I\{S > L\} . \]

(2.13)
If $\beta(x) \leq L$ then (2.12) vanishes. Otherwise, one has
\[
E[(X - x)_+ \cdot I\{S > L\}] = \alpha \cdot E[(\beta(x) - L) \cdot I(L < S \leq \beta(x))] \\
= \alpha \cdot \{E[(\beta(x) - S) \cdot I(S \leq \beta(x))] - E[(\beta(x) - S) \cdot I(S \leq L)]\} \\
= \alpha \cdot \{\pi_s[\beta(x)] - (\beta(x) - L) \cdot F_s(L) - \pi_s[L]\} \\
= \alpha \cdot \{\pi_s[\beta(x)] - \pi_s[L] + (\beta(x) - L) \cdot F_s(L)\}. 
\]

(2.14)

If $\beta(x) \leq L$ then (2.13) also vanishes. Otherwise, one has
\[
E[(X - x)_+ \cdot I\{S \leq L\}] = \alpha \cdot (\beta(x) - L) \cdot F_s(L). 
\]

(2.15)

Inserting into (2.7) one obtains
\[
\pi_x(x) = \begin{cases} 
\alpha \cdot (\beta(x) - L + \pi_s[\beta(x)] - \pi_s[L]), & \beta(x) > L, \\
0, & \beta(x) \leq L.
\end{cases} 
\]

(2.16)

Now, insert (2.11) into (2.16) to get
\[
\pi_x[Q_x(\alpha)] = \begin{cases} 
\alpha \cdot \{\pi_s[Q_s(\varepsilon)] - \pi_s[L]\}, & \beta(x) > L, \\
0, & \beta(x) \leq L.
\end{cases} 
\]

(2.17)

Finally, put (2.11) and (2.17) into (2.4), and summarize to get the desired formula. ◊

A remarkable feature of the portfolio insurance strategy is the constant amount of required economic risk capital as measured by value-at-risk and conditional value-at-risk as long as the loss probability is sufficiently small.

**Corollary 2.1.** If $\varepsilon \leq F_s(L)$ then one has
\[
CVaR_x[X] = VaR_x[X] = \frac{K(L) - L}{K(L)}. 
\]

(2.18)

**Proof.** This follows immediately from (2.5) and (2.11). ◊

It should be emphasized that in practice the condition of Corollary 2.1 is nearly always fulfilled. Even more, a relatively large range of confidence levels may be tolerated. For example, suppose the logarithm return $\ln(S_0)$ is normally distributed with mean $\mu$ and standard deviation $\sigma$. In case $L = S_0$ is “at the money”, one should have $\alpha \geq \Phi(k^{-1})$, where $k = \frac{\sigma}{\mu}$ is the coefficient of variation and $\Phi(x)$ is the standard normal distribution. Numerically, if $\mu = 0.1, \sigma = 0.2$, one has $\alpha \geq \Phi(\frac{\sigma}{\mu}) = 0.691$. An empirical study, which confirms these observations, follows in Section 3.
3. **Portfolio insurance for a market index.**

Consider portfolio insurance for the Swiss Market Index (SMI) over the one-month period between 20.11.1998 and 18.12.1998. One has \( S_0 = 7138 \) and the time horizon is \( T = \frac{1}{12} \). Following Herbert et al.(1998), p.68, the long term logarithm return \( \ln \left( \frac{S_T}{S_0} \right) \) can be assumed to be normally distributed with mean \( \mu = \left( r - \frac{1}{2} v^2 \right) T \) and volatility \( \sigma = v \cdot \sqrt{T} \). According to Hürlimann(2001e), Table 7.2, a valid parameter estimation over the one-year period between 29.9.1998 and 24.9.1999 is \( r = 0.1727, \ v = 0.2863 \). Possible exercise prices with corresponding put and call option prices as published in newspaper from 21.11.1998 are found in Table 3.1.

**Table 3.1:** put and call option prices for the SMI

<table>
<thead>
<tr>
<th>L</th>
<th>6900</th>
<th>7000</th>
<th>7100</th>
<th>7200</th>
<th>7300</th>
<th>7400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(L) )</td>
<td>154</td>
<td>187</td>
<td>218</td>
<td>272</td>
<td>406.8</td>
<td>476.8</td>
</tr>
<tr>
<td>( C(L) )</td>
<td>370</td>
<td>308</td>
<td>253</td>
<td>194</td>
<td>152.5</td>
<td>114</td>
</tr>
</tbody>
</table>

According to the put-call parity relation (2.2), the risk-free rate depends on the exercise price and is determined by

\[
\delta = \delta(L) = -\frac{1}{T} \ln \left( \frac{S_0 + P(L) - C(L)}{L} \right). \quad (3.1)
\]

The numerical percentage figures of our evaluation are summarized in Table 3.2. For \( \varepsilon \leq 0.2933 \) one sees that \( Q_\delta(\varepsilon) = S_0 \cdot \exp \left( \mu + \Phi^{-1}(\varepsilon)\sigma \right) \leq 6900 \), hence Corollary 3.1 applies.

**Table 3.2:** CVaR for portfolio insurance of SMI, \( \alpha \geq 0.7067 \)

<table>
<thead>
<tr>
<th>exercise price ( L )</th>
<th>risk-free rate ( \delta )</th>
<th>( E[R] )</th>
<th>( CVaR_\alpha[X] )</th>
<th>( RAROC_\alpha[X] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6900</td>
<td>−3.82</td>
<td>0.692</td>
<td>5.376</td>
<td>0.129</td>
</tr>
<tr>
<td>7000</td>
<td>−2.91</td>
<td>0.682</td>
<td>4.437</td>
<td>0.154</td>
</tr>
<tr>
<td>7100</td>
<td>−0.51</td>
<td>0.786</td>
<td>3.48</td>
<td>0.226</td>
</tr>
<tr>
<td>7200</td>
<td>−2.66</td>
<td>0.666</td>
<td>2.834</td>
<td>0.235</td>
</tr>
<tr>
<td>7300</td>
<td>−15.08</td>
<td>−0.441</td>
<td>3.245</td>
<td>−0.136</td>
</tr>
<tr>
<td>7400</td>
<td>−16.24</td>
<td>−0.585</td>
<td>2.821</td>
<td>−0.207</td>
</tr>
</tbody>
</table>

The CVaR risk measure is of great importance in decision making because it can be used as a tool in risk-adjusted performance measurement. Consider the random return of portfolio insurance per unit of conditional value-at-risk to a fixed confidence level \( \alpha \), called CVaR return ratio, which is defined by

\[
\frac{R}{CVaR_\alpha[-R]} \quad (3.2)
\]
where the random return $R$ has been defined in (2.3). The expected value of the CVaR return ratio measures the risk-adjusted return on capital. This way of computing the return is commonly called RAROC (e.g. Matten(1996), p.59), and is defined by

$$RAROC_{\alpha}[R] = \frac{E[R]}{CVaR_\alpha[-R]}.$$  \hfill (3.3)

Now, if an investor has to decide upon the more profitable of two portfolio insurance strategies with different exercise prices and random returns $R_1$ and $R_2$, a decision in favor of the first strategy is taken if and only if one has $RAROC_{\alpha}[R_1] \geq RAROC_{\alpha}[R_2]$ at given confidence levels $\alpha$. This preference criterion tells us that a return is preferred to another if its expected value per unit of economic risk capital is greater. In Table 3.2 the exercise price $L = 7200$ is preferred over the other ones under this RAROC criterion.

It is remarkable that the above results hold for all confidence levels $\alpha \geq 0.7067$. The CVaR measure remains also stable under variation of the volatility. Indeed, in our setting one has $Q_\alpha(\varepsilon) \leq L$ if and only if

$$r \leq \frac{1}{2} - \Phi^{-1}(\varepsilon) \cdot \frac{\varepsilon}{\sqrt{T}} + \frac{1}{T} \cdot \ln \left( \frac{L}{S_0} \right).$$ \hfill (3.4)

Since $\Phi^{-1}(\varepsilon) \leq 0$ for $\varepsilon \leq \frac{1}{2}$, the right hand side is monotone increasing in the volatility parameter. Therefore, by fixed $\varepsilon$ and $\nu$, the condition of Corollary 2.1 holds provided the one-year expected return $r$ does not exceed the value reported in Table 3.3.

**Table 3.3**: maximum percentage one-year expected return for constant CVaR by varying confidence level and volatility

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\nu$</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td></td>
<td>0.20</td>
<td>4.2</td>
<td>19.6</td>
<td>35.3</td>
<td>51.3</td>
</tr>
<tr>
<td>0.10</td>
<td></td>
<td>4.2</td>
<td>27.0</td>
<td>50.1</td>
<td>73.4</td>
<td>97.0</td>
</tr>
<tr>
<td>0.05</td>
<td></td>
<td>16.8</td>
<td>45.9</td>
<td>75.3</td>
<td>104.9</td>
<td>134.7</td>
</tr>
<tr>
<td>0.01</td>
<td></td>
<td>40.4</td>
<td>81.3</td>
<td>122.5</td>
<td>163.9</td>
<td>205.6</td>
</tr>
</tbody>
</table>

4. **Multivariate portfolio insurance.**

In general, investors do not hold a market index but a portfolio of risky assets stemming from various asset categories in different financial markets. Let $S_1, \ldots, S_m$ be the initial prices of $m$ risky assets to be held over a period of time $[0,T]$. Then $S_0 = \sum_{i=1}^{m} S_i$ is the present value of the portfolio of assets, and $w_i = \frac{S_i}{S_0}$ is the proportion of wealth invested in the $i$-th risky asset, $i = 1, \ldots, m$. Let $R_i, i = 1, \ldots, m$ be the random accumulated returns over $[0,T]$. Then the random accumulated return of the portfolio choice $w = (w_1, \ldots, w_m)$ is given and
denoted by \( R_n = \sum_{i=1}^{m} w_i R_i \). The random market value \( S = S_0 R_n \) of the portfolio at time \( T \) satisfies the relationship \( S = \sum_{i=1}^{m} X_i \), with \( X_i = S_i R_i \), \( i = 1, \ldots, m \). To evaluate CVaR following Section 2, one needs a specification of the multivariate distribution of \( R = (R_1, \ldots, R_m) \) as well as option pricing formulas corresponding to the put and call random payoffs in (2.1).

Suitable multivariate distributions with arbitrary marginals are obtained with the method of copulas. We apply the multivariate Fréchet copula constructed in Chapter VII. To maintain technicalities at a minimum level, our presentation is restricted here to the bivariate case.

With the option pricing model of Black-Scholes in mind, assume the margin \( R_i \) is lognormally distributed with parameters \( \mu_i = (r_i - \frac{1}{2} \sigma_i^2) \cdot T \) and \( \sigma_i = \gamma_i \cdot \sqrt{T} \), where \( r_i, \gamma_i \) represent the one-year expected return respectively volatility. For option pricing, we use the transformed margin \( R_i^\delta \), which is \( R_i \) but with \( r_i \) replaced by the risk-free rate \( \delta \). Then \( X_i = S_i R_i \) and \( X_i^\delta = S_i R_i^\delta \) are also lognormally distributed. The corresponding distributions are denoted \( F_i(x) \) respectively \( F_i^\delta(x) \), \( i = 1, 2 \). The bivariate distributions of the random couples \( (X_1, X_2) \) respectively \( (X_1^\delta, X_2^\delta) \) are denoted and defined by

\[
F(x_1, x_2) = C[F_1(x_1), F_2(x_2)], \quad F^\delta(x_1, x_2) = C[F_1^\delta(x_1), F_2^\delta(x_2)], \quad (4.1)
\]

where \( C(u, v) \) is the linear Spearman copula

\[
C(u, v) = (1 - |\theta|) \cdot C_0(u, v) + |\theta| \cdot C_{sgn(\theta)}(u, v), \quad \theta \in [-1, 1],
\]

\[
C_0(u, v) = uv, \quad C_1(u, v) = \min(u, v), \quad C_{-1}(u, v) = \max(u + v - 1, 0).
\]

The Spearman grade correlation coefficient and the coefficient of upper tail dependence of this copula are both equal to the dependence parameter \( \theta \). The distributions and stop-loss transforms of the dependent sums \( S = X_1 + X_2 \) and \( S^\delta = X_1^\delta + X_2^\delta \) are determined by the following analytical expressions (Theorem VII.3.1). For \( i = 1, 2 \), let \( Q_i(u) = F_i^{-1}(u), u \in [0, 1] \), be the \( u \)-quantile of \( X_i \), and set \( u_\theta = \frac{1}{2} [1 - sgn(\theta)] + sgn(\theta) u \). Then one has the formulas

\[
F_\delta [Q_1(u) + Q_2(u_\theta)] = (1 - |\theta|) \cdot F_\delta [Q_1(u) + Q_2(u_\theta)] + |\theta| \cdot u, \quad (4.3)
\]

\[
\pi_\delta [Q_1(u) + Q_2(u_\theta)] = (1 - |\theta|) \cdot \pi_\delta [Q_1(u) + Q_2(u_\theta)]
\]

\[
+ |\theta| \cdot \left\{ \pi_1 [Q_1(u)] + sgn(\theta) \pi_2 [Q_2(u_\theta)] + \frac{1}{2} [1 - sgn(\theta)] \cdot [E[X_2] - Q_2(u_\theta)] \right\}, \quad (4.4)
\]

where \( S^\perp = X_1^\perp + X_2^\perp \), with \( (X_1^\perp, X_2^\perp) \) an independent version of \( (X_1, X_2) \) such that \( X_1^\perp \) and \( X_2^\perp \) are independent and identically distributed as \( X_1, X_2 \). Similar expressions hold for \( S^\delta \) with \( Q_i(u) \) replaced by \( Q_i^\delta(u) = (F_i^\delta)^{-1}(u), i = 1, 2 \). For portfolio insurance valuation, we use the call and put option prices

\[
C(L) = e^{-\delta T} \cdot \pi_s^\delta(L), \quad P(L) = e^{-\delta T} \cdot (L + \pi_s^\delta(L)) - S_0, \quad (4.5)
\]
and conditional value-at-risk follows from the representation (2.8):
\[
CVaR_x[-R] = \frac{1}{K(L)} \left\{ K(L) - L - \left( Q_\alpha(x) - L \right) + \frac{1}{\epsilon} \left( Q_\alpha(x) + \pi_\delta \left[ Q_\delta(x) - L - \pi_\delta[L] \right] \right) \right\}. \tag{4.6}
\]

For a concrete implementation of (4.3)-(4.6), analytical expressions for one of density, distribution and stop-loss transform of the independent sums \( S^\perp = X_1^\perp + X_2^\perp \) and \( (S^\delta)^\perp = (X_1^\delta)^\perp + (X_2^\delta)^\perp \) are required. From Johnson et al. (1994), p.218, the analytical expression for the density is equal to
\[
f_{S^\perp}(x) = \frac{1}{2\pi\beta_1\beta_2} \int_0^1 \int_0^1 \exp\left\{ -\frac{1}{2} \left( \frac{\ln(1-t) + \ln(x) - \alpha_i}{\beta_i} \right)^2 - \frac{1}{2} \left( \frac{\ln(t) + \ln(x) - \alpha_2}{\beta_2} \right)^2 \right\} dt, \tag{4.7}
\]
where the parameters are given by
\[
\alpha_i = \ln(S_i) + (r_i - \frac{1}{2}v_i^2) \cdot T, \\
\beta_i = v_i \cdot \sqrt{T}, \quad i = 1,2. \tag{4.8}
\]

Assuming finite integrals are implemented, one obtains further
\[
F_{S^\perp}(x) = \int_0^x f_{S^\perp}(y) dy, \quad \pi_{S^\perp}(x) = \mu - x + \int_0^x (x - y) f_{S^\perp}(y) dy \tag{4.9}
\]

Moreover, the stop-loss transform of the margin \( X_i = S_i R_i \) reads
\[
\pi_i(x) = S_i e^{r_i T} \left( 1 - \Phi\left( \frac{\ln(x) - \alpha_i}{\beta_i} \right) \right) - x \cdot \left( 1 - \Phi\left( \frac{\ln(x) - \alpha_i}{\beta_i} \right) \right), \quad i = 1,2. \tag{4.10}
\]

For illustration, we list in Tables 4.1 to 4.3 the values of cost, CVaR and RAROC for different exercise prices by varying the dependence parameter of the bivariate return distribution. The choice of our parameters is \( \delta = \ln(1.025), \ v_1 = 0.3, v_2 = 0.2, \ S_1 = S_2 = \frac{1}{2}, \ T = 1, \ r_1 = \ln(1.15), \ r_2 = \ln(1.10) \). For all \( \varepsilon \leq 0.2 \) (or \( \alpha \geq 0.8 \)) one has
\[
Q_\delta(x) \leq Q_\delta(x) + Q_\delta(x) = \frac{1}{2} \left\{ \exp(\mu_1 + \Phi^{-1}(\varepsilon)\sigma_1) + \exp(\mu_2 + \Phi^{-1}(\varepsilon)\sigma_2) \right\} \leq 0.88263 . \tag{4.11}
\]
It follows that Corollary 3.1 applies whenever \( L \geq 0.9 \).
Applying the RAROC criterion, the exercise price \( L = 1.1 \) is preferred. Moreover, in accordance with the usual standards in finance, low dependence between returns is also preferred. Again, all these results hold under the weak assumption \( \alpha \geq 0.8 \).
### Table 4.1: Cost of bivariate portfolio insurance

<table>
<thead>
<tr>
<th>θ</th>
<th>L</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1.02303</td>
<td>1.05962</td>
<td>1.11735</td>
</tr>
<tr>
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<td></td>
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<tr>
<td>1</td>
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<td>1.04445</td>
<td>1.08644</td>
<td>1.14384</td>
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</tbody>
</table>

### Table 4.2: CVaR for bivariate portfolio insurance

<table>
<thead>
<tr>
<th>θ</th>
<th>L</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
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<td>0.12026</td>
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<td>0.01553</td>
</tr>
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<td>0.25</td>
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<td>0.12484</td>
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<td>0.02133</td>
</tr>
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<td>1</td>
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</table>

### Table 4.3: RAROC for bivariate portfolio insurance

<table>
<thead>
<tr>
<th>θ</th>
<th>L</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
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CHAPTER IX. GUARANTEED CASH-FLOWS UNDER FRECHET-MARKOV RETURN MODELS

1. Introduction.

The present Chapter is devoted to the evaluation of the multi-period economic risk capital (ERC) of a guaranteed cash-flow under two simple bi- and triatomic Fréchet-Markov models of the return. As seen in Chapter II, the economic risk capital associated to some future random loss can be identified with the conditional value-at-risk (CVaR) of the loss given some confidence level $\alpha$. Mathematically, this is defined as expected value of the “$\alpha$-tail transform”, which is viewed as a convex combination of the usual value-at-risk (VaR) and the upper conditional value-at-risk (CVaR+). A more detailed outline of the content follows.

Section 2 introduces our multi-period ERC valuation model for guaranteed discrete-time cash-flows under random returns. To protect the random cash-flow against adverse returns below a deterministic guaranteed return, an investor is supposed to buy in each period a put option. This generates some cost of the guarantee at the finite time horizon. The cash-flow risk is defined to be the negative of the possible random gain generated by the protected cash-flow in excess of the sum of the guaranteed cash-flow value and the cost of the guarantee. The multi-period ERC of a guaranteed cash-flow is set equal to the CVaR of the cash-flow risk given some confidence level. A main application concerns the important notion of risk-adjusted return on capital (RAROC) of a guaranteed cash-flow. Used as a tool in risk-adjustment performance measurement, the RAROC criterion tells us when a guaranteed cash-flow under random returns is preferred to another one.

In Section 3 we consider the following biatomic Markov chain return model. In each period the random return is uniquely identified as a biatomic random variable with fixed mean, standard deviation and skewness. The random returns in the multi-period of valuation are assumed to be identically distributed and follow a one-parameter Markov chain with Fréchet copula bivariate dependence structure, called for short Fréchet-Markov model, which includes as extreme cases independent and comonotone returns. The cost of guarantee is determined under the usual risk-neutral valuation method. Then, it is shown how to evaluate numerically the CVaR risk measure.

Section 4 contains a detailed study of the often encountered guaranteed annuity-due with constant payments under our biatomic Fréchet-Markov return model. A remarkable feature of the put option strategy used to protect the random annuity is the constant amount of ERC as long as the chosen confidence level is sufficiently high. Since ERC is bounded above by the cost of guarantee, we maximize this quantity under variation of the skewness parameter. This yields prudent ERC upper bounds. The obtained results are illustrated and discussed at the real life example of the Fed Funds rates of return over the period 1988-1997.

Section 5 extends the analysis using a similar triatomic Fréchet-Markov chain return model. Based on the solution of the algebraic moment problem of order three, we first identify the one-period random returns as a one-parameter family of triatomic random variables with fixed mean, standard deviation, skewness and kurtosis. The remaining parameter is uniquely determined by the VaR condition that a return less than the smallest possible atom occurs only with the loss probability $\varepsilon = 1 - \alpha$, where $\alpha$ is the confidence level used to calculate CVaR. To determine the cost of guarantee, we replace the probabilities by risk-neutral probabilities. The numerical evaluation of the CVaR risk measure is similar to the one presented in Section 3. A comparative numerical illustration follows for the guaranteed annuity-due with constant payments.
2. Multi-period ERC valuation model.

A discrete-time cash-flow over the time horizon \([0,T]\), where a positive integer, is a sequence of payments \(c = (c_1, \ldots, c_T)\), where the payment \(c_k\) is due at time \(k - 1, k = 1, \ldots, T\). The payments may be strictly positive (long position), strictly negative (short position) or zero (vanishing cash payment).

For simplicity suppose that the return in the \(k\)-th period \([k - 1, k]\) is a random variable \(R_k, k = 1, \ldots, T\). We assume that \(R_1, R_2, \ldots, R_T\) are identically distributed with finite expected return \(r = E[R_k]\) and standard deviation \(\sigma = \sqrt{\text{Var}[R_k]}\). Over the time horizon \([0,T]\), the payments should yield the deterministic guaranteed return \(r_g\). The corresponding guaranteed cash-flow induces a liability, whose accumulated value at time \(T\) is given by

\[
L_T = \sum_{k=1}^{T} c_k \cdot (1 + r_g)^{T-k+1}.
\]  

(2.1)

To obtain the guaranteed return with certainty, an investor will buy on the financial market in each period \([k - 1, k]\) a certain amount of put options with payoff \((r_g - R_k)\). According to the put-call parity relation, the total random return in period \([k - 1, k]\) satisfies the identity

\[
R_k + (r_g - R_k) = r_g + (R_k - r_g). 
\]  

(2.2)

The random accumulated value at time \(T\) of the protected cash-flow is then given by

\[
V_T = \sum_{k=1}^{T} c_k \cdot \prod_{j=k}^{T} [1 + r_g + (R_k - r_g)].
\]  

(2.3)

It remains to determine the cost of guarantee. Denote by \(P(r_g)\) the put option price, to be paid at time \(k - 1\), which yields in each period \([k - 1, k]\) the payoff \((r_g - R_k)\), \(k = 1, \ldots, T\). In each period \([t - 1, t], t = 1, \ldots, T\), the accumulated amount of the sequence of payments \(c_1, \ldots, c_t\), that is \(\sum_{k=1}^{t} c_k (1 + r_g)^{-k+1}\), must be guaranteed at time \(t\) by buying put options with payoff \((r_g - R_k)\). Using (2.2) the cost of this guarantee at time \(t - 1\) equals \(K_{t-1} = P(r_g) \cdot \sum_{k=1}^{t} c_k (1 + r_g)^{-k+1}, t = 1, \ldots, T\). Valued at the risk-free return \(r_f\), the accumulated value at time \(T\) of the total cost of guarantee is then equal to

\[
C_T = \sum_{t=0}^{T-1} K_t (1 + r_f)^{T-t}.
\]  

(2.4)

It will be assumed throughout that the risk-free return, the guaranteed return and the expected return satisfy the inequality

\[
r_g < r_f < r.
\]  

(2.5)
The fact that \( r_g < r_f \) follows from diverse arguments (e.g. Devolder(1986/91), Kozik(1991), Hürlimann(1991a/b/96b), Wilkie(1991). The above protected cash-flow should be compared with the guaranteed cash-flow under the cost of guarantee. The generated aggregate surplus at time \( T \), denoted as difference between assets and liabilities, is denoted and equal to

\[
G_T = V_T - L_T - C_T. \tag{2.6}
\]

It represents the possible random gain of the protected cash-flow in excess of the sum of the guaranteed cash-flow value and the cost of guarantee. Its negative value is called cash-flow risk. The economic risk capital (ERC) associated to the cash-flow risk \( X_T = -G_T = C_T + L_T - V_T \) will be identified with the conditional value-at-risk to some confidence level \( \alpha \) as defined in Chapter II.

In the special situation of discrete loss distributions with ordered support \( x_1 < x_2 < \ldots < x_k < x_{k+1} < \ldots \), which will be used to evaluate CVaR bounds in the present Chapter, numerical evaluation proceeds as follows. Let \( f_k = \Pr(X = x_k) \) denote the probability that the loss takes the value \( x_k \), \( k = 1, 2, 3, \ldots \), and assume the finite mean \( \mu_X = E[X] \) is known. Determine the unique index \( k_\alpha \) such that

\[
\sum_{k=1}^{k-1} f_k < \alpha \leq \sum_{k=1}^k f_k. \tag{2.7}
\]

Then one has

\[
VaR_\alpha[X] = x_{k_\alpha},
\]

and one obtains from formula (II.2.7) that

\[
CVaR_\alpha[X] = VaR_\alpha[X] + \frac{1}{\epsilon} \cdot \left\{ \mu_X - VaR_\alpha[X] + E[\left( VaR_\alpha[X] - X \right)] \right\}
\]

\[
= \frac{1}{\epsilon} \cdot \left\{ \mu_X - \alpha \cdot x_{k_\alpha} + \sum_{k=0}^k (x_{k_\alpha} - x_k) \cdot f_k \right\}. \tag{2.8}
\]

In particular, the loss probabilities must only be evaluated up to the index \( k_\alpha \) satisfying the inequality (2.7).

The CVaR risk measure is of great importance in decision making because it can be used as a tool in risk-adjusted performance measurement. Consider the random gain (2.6) per unit of conditional value-at-risk, called CVaR gain ratio, which is defined by

\[
\frac{G_T}{CVaR_\alpha[X_T]}. \tag{2.9}
\]

The expected value of the CVaR gain ratio measures the risk-adjusted return on capital. This way of computing the return is called RAROC (e.g. Matten(1996), p.59), and is defined by

\[
RAROC_\alpha[G_T] = \frac{E[G_T]}{CVaR_\alpha[X_T]}. \tag{2.10}
\]
Now, if an investor has to decide upon the more profitable of two protected cash-flows with associated random gains \( G_1^t \) and \( G_2^t \), a decision in favor of the first one is taken if and only if \( \text{RAROC}_a[G_1^t] \geq \text{RAROC}_a[G_2^t] \) at given confidence levels \( \alpha \). The RAROC criterion tells us that a random gain is preferred to another if its expected value per unit of economic risk capital is greater.

A simple alternative to RAROC is the inverse coefficient of variation (ICV) of the random gain defined by

\[
\text{ICV}[G_t] = \frac{1}{\text{CV}[G_t]}, \quad \text{CV}[G_t] = \frac{\text{Var}[G_t]}{E[G_t]},
\]

which represents the expected value of the random gain per unit of standard deviation. The ICV criterion tells us that \( G_1^t \) is preferred to \( G_2^t \) if and only if \( \text{ICV}[G_1^t] \geq \text{ICV}[G_2^t] \) or equivalently \( \text{CV}[G_1^t] \leq \text{CV}[G_2^t] \).


Without further specification of the identically distributed returns \( R_1, R_2, ..., R_T \), the ERC and RAROC quantities cannot be calculated. Besides the mean \( r \) and standard deviation \( \sigma \), we will assume that the skewness of the random return \( R_k \) equals \( \gamma \), \( k = 1, ..., T \). By assuming further biatomic returns, these characteristics uniquely determine \( R_k \).

**Proposition 3.1.** The support \( \{d, u\}, d < u \), and probabilities \( \{p, q = 1 - p\} \) of a biatomic random variable with mean \( r \), standard deviation \( \sigma \) and skewness \( \gamma \) are uniquely determined by

\[
d = r - \frac{1}{2} \sigma \left( \sqrt{4 + \gamma^2} - \gamma \right),
\]
\[
u = r + \frac{1}{2} \sigma \left( \sqrt{4 + \gamma^2} + \gamma \right),
\]
\[
p = \frac{1}{2} \left( 1 + \frac{\gamma}{\sqrt{4 + \gamma^2}} \right)
\]

**Proof.** The atoms of a standardized biatomic random variable with mean \( r = 0 \), standard deviation \( \sigma = 1 \) and skewness \( \gamma \) solve the algebraic moment problem of order two, that is the non-linear equations

\[
pd + qu = 0,
\]
\[
pd^2 + qu^2 = 1,
\]
\[
pd^3 + qu^3 = \gamma.
\]

According to Mammana(1954) (see Hürlimann(1998e), chapter I, or Appendix C), the atoms \( d, u \) are the distinct real zeros of the standard quadratic orthogonal polynomial of degree two

\[
p_2(x) = x^2 - px - 1,
\]

that is
\[ d = \frac{1}{2} \left[ \gamma - \sqrt{4 + \gamma^2} \right] \quad u = \frac{1}{2} \left[ \gamma + \sqrt{4 + \gamma^2} \right] \]

The formulas (3.1) follow immediately. \( \Diamond \)

Let us first calculate the cost of the guarantee (2.4). Following the usual risk-neutral valuation (e.g. Cox and Ross (1976)), the put option price with payoff \( (r_g - R_k)_+ \) equals

\[
P(r_g) = \frac{1}{1 + r_f} E^*[(r_g - R_k)_+] \\
= \frac{1}{1 + r_f} \cdot \{(r_g - d)_+ \cdot p^* + (r_g - u)_+ \cdot (1 - p^*)\} 
\]

where the risk-neutral probability, which satisfies the condition \( E^*[R_g] = r_f \), is given by

\[
p^* = \frac{u - r_f}{u - d}, \quad d < r_f < u. \quad (3.3)\]

The dependence structure between the returns \( R_1, R_2, ..., R_T \) must also be specified. For simplicity, we assume the Markov chain property

\[
\Pr\left(R_k = x_k | R_j = x_j, \ j = 1, ..., k - 1 \right) = \Pr\left(R_k = x_k | R_{k-1} = x_{k-1} \right), \quad k = 2, ..., T, \quad (3.4)
\]

where the \( x_k \)'s belong to the supports of the \( R_k \)'s. It follows that the joint probability function can be calculated from the formula

\[
\Pr\left(R_k = x_k, k = 1, ..., T \right) = \Pr(R_k = x_k) \cdot \prod_{j=2}^{T} \Pr\left(R_j = x_j | R_{j-1} = x_{j-1} \right) \quad (3.5)
\]

The conditional probabilities

\[
\Pr\left(R_j = x_j | R_{j-1} = x_{j-1} \right) = \frac{\Pr(R_{j-1} = x_{j-1}, R_j = x_j)}{\Pr(R_{j-1} = x_{j-1})} \quad (3.6)
\]

depend upon the knowledge of a bivariate distribution for the random couples \( (R_{j-1}, R_j), j = 2, ..., T, \) which are assumed to be identically distributed. In general, given a random couple \( (X,Y) \) with finite discrete support \( \{(x_i, y_i), i = 1, ..., m, k = 1, ..., n\} \), marginals \( F_i = \Pr(X \leq x_i), \ G_k = \Pr(Y \leq y_k) \), and joint distribution \( H_{ik} = \Pr(X \leq x_i, Y \leq y_k) \), we will assume that the latter follows a Fréchet copula such that for some \( \theta \in [0,1] \):

\[
H_{ik} = (1 - \theta) F_i G_k + \theta \min\{F_i, G_k\}. \quad (3.7)
\]
This one-parameter positive quadrant dependent statistical model includes independence \( (\theta = 0) \) and comonotonocity \( (\theta = 1) \). In our situation, the couples \( (R_{j-1}, R_j), j = 2, ..., T \), have support \( \{(d, d), (d, u), (u, d), (u, u)\} \) and probabilities \( \{p_{11}, p_{12}, p_{21}, p_{22}\} \) such that

\[
(p_q) = \begin{pmatrix}
p(p + \theta q) & (1 - \theta) pq \\
(1 - \theta) pq & q(q + \theta p)
\end{pmatrix},
\]

where \( d, u, p, q = 1 - p \) have been defined in (3.1), and \( \theta \in [0, 1] \). The simple model defined by (3.5), (3.6) and (3.8) is called biatomic Fréchet-Markov return model. The independent case \( \theta = 0 \) is called biatomic random walk return model while the comonotone case \( \theta = 1 \) is called biatomic comonotone return model.

To evaluate the random variable (2.3), one observes that

\[
V_T = \sum_{k=1}^{T} c_k \prod_{j=k}^{T} Y_k, \text{ or recursively}
V_k = Y_k(V_{k-1} + c_k), \quad k = 2, ..., T,
\]

where \( Y_k = 1 + r_g + (R_k - r_g), k = 1, ..., T \), is a finite sequence of Fréchet-Markov identically distributed biatomic random variables with supports \( \{a, b\}, a < b \), and probabilities \( \{p, q = 1 - p\} \) given by (assume \( r_g \geq d = r - \frac{1}{2} \sigma(\sqrt{4 + \gamma^2} - \gamma) \))

\[
a = 1 + r_g,
b = 1 + r + \frac{1}{2} \sigma(\sqrt{4 + \gamma^2} + \gamma),
p = \frac{1}{2} \left(1 + \frac{\gamma}{\sqrt{4 + \gamma^2}}\right).
\]

To obtain the \( n = 2^T \) atoms of the discrete random variable \( V_T \), consider the set \( \Delta \) of all zero-one vectors \( \delta = (\delta_1, ..., \delta_T) \) with \( \delta_k \in \{0, 1\} \). Then each \( \delta \in \Delta \) yields an atom

\[
v_{\delta} = \sum_{k=1}^{T} c_k \prod_{j=k}^{T} a^{1-\delta_j} b^{\delta_j},
\]

with probability

\[
h_{\delta} = \Pr(V_T = v_{\delta}) = p^{1-\delta_1}q^{\delta_1} \prod_{k=2}^{T} \left[(1 - \theta) \cdot p^{1-\delta_k} q^{\delta_k} + \theta \cdot \epsilon(\delta_{k-1}, \delta_k)\right],
\]

where \( \epsilon(x, y) = 1 \) if \( y = x \) and 0 else. To evaluate CVaR of the cash-flow risk \( X_T = C_T + L_T - V_T \), reorder and rename the \( n \) atoms \( x_\delta = C_T + L_T - v_\delta \) of \( X_T \) in ascending order such that \( x_1 < x_2 < ... < x_n \) and \( f_k = \Pr(X_T = x_k), k = 1, ..., n \). Using (2.6) one obtains the formula
\[ \text{CVaR}_\alpha[X_T] = x_{k_\alpha} + \frac{1}{1 - \alpha} \sum_{k=k_\alpha}^{n} (x_k - x_{k_\alpha}) \cdot f_k, \]  

(3.13)

where \( k_\alpha \) satisfies the inequality (2.7).

4. Guaranteed annuity-due with constant payments.

The useful special case \( c_k = 1, k = 1, \ldots, T \), represents a guaranteed annuity-due with \( T \) annual payments of amount \( 1 \). The accumulated value at time \( T \) of a deterministic annuity-due with constant payments of amount \( 1 \) and interest rate \( j \) is denoted by

\[ S(T, j) = \left( \frac{1 + j}{j} \right)^T - 1. \]  

(4.1)

The protected random annuity with guaranteed interest rate \( r_g \) induces the deterministic liability at time \( T \) of amount

\[ L_r = S(T, r_g) \]  

(4.2)

and the cost of guarantee (use formula (2.4))

\[ C_r = P(r_g) \sum_{t=1}^{T} S(t, r_g)(1 + r_j)^{T-t+1} \]

\[ = P(r_g) \left( \frac{1 + r_g}{r_g} \right)^{T} \left( 1 + \frac{r_g}{1 + r_g} S(T, r_g) - S(T, r_j) \right). \]  

(4.3)

In general, the mean \( \mu[V_r] \) and variance \( \sigma^2[V_r] \) of \( V_r \) can be calculated numerically using (3.11) and (3.12). For the important biatomic random walk return model, that is the independent case \( \theta = 0 \), there exists even the explicit expressions (see Burnecki et al.(2001), Corollary 3.1, which corrects some main results in Zaks(2001)):

\[ \mu[V_r] = S(T, r_g), \quad r_g = E[Y_k] - 1 = ap + bq - 1, \]  

(4.4)

\[ \sigma^2[V_r] = 2(1 + r_j)^{T+1} S(T, g) - (2 + r_j)S(T, f) - (1 + r_j)S(2T, r_j) + 2(1 + r_j)S(T, r_j) \]  

\[ = \frac{r_g}{r_j}. \]  

(4.5)

where one sets

\[ f = 2r_j + r_j^2 + s_j^2, \quad s_j^2 = Var[Y_k] = (b-a)^2 pq, \quad g = r_g + \frac{s_j^2}{1 + r_g}. \]  

(4.6)

As a general consequence, the inverse coefficient of variation of gain is explicitly given by

\[ \text{ICV}[G_r] = \frac{\mu[V_r] - L_r - C_r}{\sigma[V_r]} . \]  

(4.7)
The calculation of CVaR is also rather explicit. First of all, it is convenient to number the zero-one vectors \( \delta \in \Delta \) in such a way that \( \delta_{k-1} = (\delta_{k-1,1}, \ldots, \delta_{k-1,T}) \) precedes \( \delta_k = (\delta_{k,1}, \ldots, \delta_{k,T}) \) in the lexicographic order, \( k = 2, \ldots, n = 2^T \). Since \( a < b \) one sees immediately that \( v_{\delta_{k-1}} < v_{\delta_k}, k = 2, \ldots, n \). Setting \( x_k = C_T + L_T - v_{\delta_{k-1}} \) and \( f_k = h_{\delta_{k-1}}, k = 1, \ldots, n \), the atoms of \( X_T \) form automatically an increasing sequence and the proposed numerical algorithm to evaluate CVaR applies.

A remarkable feature of the put option strategy used to protect the random annuity is the constant amount of required economic risk capital as long as the loss probability is sufficiently small. In our special situation, this constant is the cost of guarantee. As a consequence, the economic risk capital is always bounded above by the cost of guarantee.

**Proposition 4.1.** If \( \varepsilon < p(p + \theta q)^{T-1}, \theta \in [0,1] \), \( r_g \geq d = r - \frac{1}{2} \sigma(\sqrt{4 + \gamma^2} - \gamma) \), one has

\[
CVaR_a[X_T] = VaR_a[X_T] = C_T. \tag{4.8}
\]

**Proof.** For \( \delta_1 = (0, \ldots, 0) \) one has \( v_{\delta_1} = S(T, a - 1) \) and \( h_{\delta_1} = p(p + \theta q)^{T-1} \). Since \( \varepsilon < f_n = p(p + \theta q)^{T-1} \) the inequality (2.7) is only satisfied when \( k_n = n \), hence

\[
CVaR_a[X_T] = VaR_a[X_T] = x_n = C_T + L_T - v_{\delta_1}. \tag{4.8}
\]

The formulas (4.8) follows using (4.2) and (4.3) and the fact that \( a = 1 + r_g \) under the assumption \( r_g \geq d \). ◊

The quantity RAROC satisfies a similar property.

**Corollary 4.1.** If \( \varepsilon < p(p + \theta q)^{T-1}, \theta \in [0,1] \), \( r_g \geq d = r - \frac{1}{2} \sigma(\sqrt{4 + \gamma^2} - \gamma) \), one has

\[
RAROC_a[G_T] = \frac{\mu[V_T] - S(T, r_g) - C_T}{C_T}. \tag{4.9}
\]

**Example 4.1.**

It is interesting to look at the extreme case of the biatomic comonotone return model obtained when \( \theta = 1 \). From (3.12) one sees that \( h_\delta = 0 \) unless \( \delta = (0, \ldots, 0) \) or \( \delta = (1, \ldots, 1) \). In the special case of constant payments, one obtains a biatomic random value \( V_T \) with support \( \{S(T, a - 1), S(T, b - 1)\} \) and probabilities \( \{p, q = 1 - p\} \). Then the cash-flow risk \( X_T \) is also biatomic with support \( \{x_1, x_2\} = \{C_T + L_T - S(T, b - 1), C_T + L_T - S(T, a - 1)\} \) and probabilities \( \{f_1, f_2\} = \{q, p\} \). The weighted average representation of CVaR yields in the biatomic case (use formula (II.2.6)) :

\[
CVaR_a[X_T] = \lambda x_1 + (1 - \lambda)x_2, \quad \lambda = \begin{cases} \frac{\varepsilon - a}{1-a}, & \varepsilon \geq p, \\ 0, & \varepsilon < p. \end{cases} \tag{4.10}
\]

If \( \varepsilon < p \) one recovers the assertions of Proposition 4.1 and Corollary 4.1. As a remarkable feature we show that this condition is almost always fulfilled.
Corollary 4.2. If \( \theta = 1, \; \varepsilon < \frac{1}{2}, \; \gamma > -\sqrt{\frac{2}{\varepsilon}} \), one has \( CVaR_\alpha[X_T] = VaR_\alpha[X_T] = C_T \).

**Proof.** Recall from (3.1) that \( p = \frac{1}{2}(1 + \frac{\gamma}{\sqrt{4+\gamma^2}}) \) with \( \gamma \) the skewness. If \( \gamma > 0 \) one has \( p > \frac{1}{2} > \varepsilon \), hence (4.11) holds. If \( \gamma < 0 \) the condition \( \varepsilon < p \) is equivalent with \( \gamma^2 < \frac{1}{\varepsilon} \), and is fulfilled by assumption on \( \gamma \). \( \diamond \)

In general, from Proposition 4.1 it follows that there is a maximum time horizon for which \( CVaR = VaR \) is maximum. For the confidence \( \alpha = 0.99 \), Table 4.1 lists the maximum time

\[
T_{\text{max}} = 1 + \left[ \frac{\ln(\varepsilon) - \ln(p)}{\ln(p + \theta q)} \right], \; \varepsilon = 1 - \alpha, \tag{4.11}
\]

in function of the skewness \( \gamma \) and the dependence parameter \( \theta \).

**Table 4.1**: maximum time horizon for maximum CVaR

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<td>17</td>
<td>20</td>
<td>26</td>
<td>35</td>
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<td>107</td>
</tr>
<tr>
<td>0.75</td>
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<td>42</td>
<td>63</td>
<td>128</td>
</tr>
<tr>
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<td>15</td>
<td>18</td>
<td>20</td>
<td>24</td>
<td>29</td>
<td>37</td>
<td>50</td>
<td>76</td>
<td>153</td>
</tr>
</tbody>
</table>

Since ERC is bounded above by the cost of guarantee, it is interesting to analyze when this quantity is maximal. This happens in case the put option price \( P(r_g) \) is maximal. We determine the maximum put option price as function of the skewness parameter \( \gamma \). In general, replacing \( P(r_g) \) by its maximal value yields the maximum cost of guarantee under the biatomic Fréchet-Markov return model. Since CVaR is translation-invariant (property of a coherent risk measure), one has \( CVaR_\alpha[X_T] = C_T + L_T - CVaR_\alpha[-V_T] \). Therefore, using the maximum cost of guarantee, CVaR is replaced by a prudent upper bound.

**Proposition 4.2.** Suppose that \( r_g < r_f < r, \; (r_f - r_g)(r - r_g) < \sigma^2 \), and that the returns \( R_k, k = 1, ... , T \) form an identically distributed sequence of biatomic random variables with mean \( r \), standard deviation \( \sigma \), and arbitrary skewness \( \gamma \). Then the put option price (3.2) with payoff \( (r_f - r_k) \), given by

\[
P(r_g) = -\frac{1}{2}(r_g - r_f) + \frac{\sigma^2 - (r_f - r_g)(r - r_f)}{\sigma \sqrt{4 + \gamma^2}} - \frac{1}{2} \frac{(2r_f - r_g)}{\sqrt{4 + \gamma^2}}, \; d \leq r_g \leq u, \tag{4.12}
\]

is maximal at the skewness parameter
\[
\gamma_0 = -2\sigma \cdot \frac{(r - r_f) + (r - r_g) - (r - r_f)(r - r_g) + \sigma^2}{(r - r_g)(r - r_f) + \sigma^2} < 0. \tag{4.13}
\]

**Proof.** Under the restriction \( d \leq r_g \leq u \) one finds using (3.1)-(3.3) the expression (4.12) for the put option price. View this as a function \( f(\gamma) \) of the skewness parameter. One has

\[
f'(\gamma) = -\frac{A\gamma + 4B}{(4 + \gamma^2)^2}, \quad A = \sigma^2 - (r - r_f)(r - r_g), \quad B = \frac{1}{2}(2r - r_f - r_g).
\]

The stationary point \( \gamma_0 = -\frac{4B}{A} \), which is equal to (4.13), yields a maximum because \( f''(\gamma_0) = -\frac{A}{8(1 + 4\gamma^2)^3} < 0 \). With the maximizing value \( \gamma = \gamma_0 \), it is not difficult to show that the condition \( d \leq r_g \leq u \) is fulfilled provided \( (r - r_f)(r - r_g) < \sigma^2 \). The result is shown. \( \diamond \)

Table 4.2 illustrates our findings at the specific biatomic random walk return model with \( \theta = 0 \) and parameters \( r_g = 4.25\% < r_f = 5\% < r = 5.81\% \), \( \sigma = 1.9558\% \), \( \gamma = 0.3032 \). Similar calculations could be done for other values \( \theta > 0 \) of the dependence parameter. These values of \( r, \sigma, \gamma \), which are borrowed from Das(2002), Table 1, p.29, are the summary statistics for the Fed Funds rate over the period January 1988 to December 1997. The specification (3.10) of the random returns \( y_k \), required for the evaluation of the random annuity value \( V_T \), is done using these values. The “guaranteed” biatomic return \( y_k - 1 \) jumps between the values \( a - 1 = 4.25\% \) and \( b - 1 = 8.085\% \). With the observed skewness the return \( R_k \) jumps between \( 4.128\% \) and \( 8.085\% \). Compare this with the observed range, which varies between the minimum rate \( 2.58\% \) and the maximum rate \( 10.71\% \). With the skewness \( \gamma_0 = -3.61909 \), which maximizes the cost of guarantee after Proposition 4.3, the obtained range between \( -1.773\% \) and \( 6.314\% \) is certainly too much negatively skewed. A compromise for the calculation of \( P(r_g) \) is to take \( \gamma = -1.046 \) for which \( R_k \) jumps between \( 2.58\% \) and \( 6.994\% \). This skewness value is used for a prudent estimation of the cost of guarantee in Table 4.2. For \( T > 8 \) there is a considerable decrease in ERC per unit of liability. This effect is a consequence of the chosen confidence level.

**Table 4.2**: ERC and RAROC of a guaranteed annuity-due

<table>
<thead>
<tr>
<th>T</th>
<th>( L_T )</th>
<th>( 100 \cdot C_T )</th>
<th>( 100 \cdot CVaR )</th>
<th>( \mu[V_T] )</th>
<th>( \sigma[V_T] )</th>
<th>( ICV[G_T] )</th>
<th>RAROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.043</td>
<td>0.75</td>
<td>0.75</td>
<td>1.059</td>
<td>0.019</td>
<td>0.445</td>
<td>1.072</td>
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<td>0.598</td>
<td>1.078</td>
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<td>1.54</td>
<td>3.367</td>
<td>0.076</td>
<td>0.714</td>
<td>1.084</td>
</tr>
<tr>
<td>4</td>
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<td>1.94</td>
<td>1.94</td>
<td>4.624</td>
<td>0.116</td>
<td>0.813</td>
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<tr>
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<td>5.675</td>
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<td>2.35</td>
<td>5.954</td>
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<td>0.901</td>
<td>1.096</td>
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<td>6</td>
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<td>2.77</td>
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<td>0.217</td>
<td>0.978</td>
<td>1.103</td>
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<tr>
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<td>3.19</td>
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<td>0.279</td>
<td>1.053</td>
<td>1.109</td>
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<td>8</td>
<td>9.692</td>
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<td>3.62</td>
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<td>9</td>
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<td>12.107</td>
<td>0.429</td>
<td>1.184</td>
<td>1.165</td>
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<tr>
<td>10</td>
<td>12.662</td>
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<td>4.10</td>
<td>13.877</td>
<td>0.518</td>
<td>1.244</td>
<td>1.241</td>
</tr>
</tbody>
</table>
5. Triatomic Fréchet-Markov return model.

Suppose that the identically distributed returns \( R_k, k = 1, \ldots, T \), have the mean \( r \), standard deviation \( \sigma \), skewness \( \gamma \) and kurtosis \( \gamma_2 \). If one assumes triatomic returns, these characteristics define the following one-parameter family of returns \( R_k \).

**Proposition 5.1.** Suppose that \( \Delta = \gamma_2 - \gamma^2 + 2 \geq 0 \). The support \( \{x_1, x_2, x_3\} \), \( x_1 < x_2 < x_3 \), and probabilities \( \{p_1, p_2, p_3\} \) of a triatomic random variable with mean \( r \), standard deviation \( \sigma \), skewness \( \gamma \) and kurtosis \( \gamma_2 \), are determined as follows:

\[
x_1 = r + \sigma c \in (-\infty, r + \sigma c], \quad c = \frac{1}{2} \left( \gamma - \sqrt{4 + \gamma^2} \right),
\]

\[
x_2 = r + \sigma \varphi(x, \psi(x)) \in \left[ r + \sigma c, r + \sigma c^2 \right], \quad \overline{c} = \frac{1}{2} \left( \gamma + \sqrt{4 + \gamma^2} \right),
\]

\[
x_3 = r + \sigma \psi(x) \in \left[ r + \sigma c^2, \infty \right),
\]

\[
p_i = p \left( \frac{x_i - r}{\sigma} \right), \quad i = 1, 2, 3,
\]

where the functions \( \varphi(u, v), \psi(u) \) and \( p(u) \) are defined by

\[
\varphi(u, v) = \frac{\gamma - u - v}{1 + uv},
\]

\[
\psi(u) = \frac{1}{2} \left( \frac{C(u) - \sqrt{C(u)^2 + 4q(u)}D(u)}{q(u)} \right),
\]

\[
C(u) = \gamma q(u) + \Delta u, \quad D(u) = \Delta + q(u),
\]

\[
q(u) = 1 + \mu - u^2, \quad \Delta = 2 + \gamma_2 - \gamma^2,
\]

\[
p(u) = \frac{\Delta}{q(u)^2 + \Delta (1 + u^2)}.
\]

**Proof.** It suffices to look at standardized triatomic random variables with atoms \( z_i = \frac{x_i - r}{\sigma} \), \( i = 1, 2, 3 \), for which the result follows from Proposition C.2.2. \( \diamond \)

To perform our evaluations, a specification of \( R_k \) using Proposition 5.1 must be made. We suggest to choose \( R_k \) such that a return less than \( x < r + \sigma c \) occurs only with the loss probability \( \varepsilon = 1 - \alpha \), where \( \alpha \) is the confidence level used to calculate CVaR. Then \( x \) is uniquely determined by the condition

\[
p(x) = \varepsilon, \quad x \in (-\infty, c].
\]

The cost of guarantee (2.4) is in great part determined by the put option price

\[
P(r_g) = \frac{1}{1 + r_f} E \left[ (r_g - R_g) \right].
\]

To fulfill the risk-neutral probability condition \( E^*[R_g] = r_f \), we
suggest to replace the probabilities \( p_i \) by risk-neutral probabilities \( p_i^* = p \left( \frac{x_i^* - r}{\sigma} \right) \), with \( x_1^* = r + \alpha x^* \), \( x_2^* = r + \sigma \varphi(x^*, \psi(x^*)) \), \( x_3^* = r + \sigma \psi(x^*) \). Explicitly, \( x^* \) solves the equation
\[
[r + \alpha x^*] p(x^*) + [r + \sigma \varphi(x^*, \psi(x))] p(\varphi(x^*, \psi(x^*))) + [r + \sigma \psi(x^*)] p(\psi(x^*)) = r_f . \tag{5.7}
\]
The put option price is then given by
\[
P(r_g) = \frac{1}{1 + r_f} \left\{ \left( r_g - r - \alpha x^* \right) p(x^*) + \left( r_g - r - \sigma \varphi(x, \psi(x)) \right) p(\varphi(x^*, \psi(x^*))) \right\} . \tag{5.8}
\]
Assuming the Markov chain property (3.4), it remains to specify the dependence structure between the random couples \( \{R_{j-1}, R_j\}, j = 2, \ldots, T \), which are again assumed to be identically distributed. As in Section 3, these couples follow Fréchet copulas and have support \( \{(x_i, x_j), 1 \leq i, j \leq 3\} \) and probabilities \( \{p_{ij}, 1 \leq i, j \leq 3\} \) such that
\[
p_{ij} = \begin{cases} p_i [p_i + \theta (1 - p_i)], & j = i, \\ (1 - \theta) p_i p_j, & j \neq i, \end{cases} \tag{5.9}
\]
where the \( x_i \)'s and \( p_i \)'s have been defined in Proposition 5.1, and \( \theta \in [0,1] \). This simple model is called triatomic Fréchet-Markov return model. The independent case \( \theta = 0 \) is called triatomic random walk return model while the comonotone case \( \theta = 1 \) is called triatomic comonotone return model.

To evaluate (2.3) we use the representation (3.4), where \( Y_k, k = 1, \ldots, T \), is a finite sequence of Fréchet-Markov identically distributed triatomic random variables with supports \( \{a_1, a_2, a_3\}, a_1 < a_2 < a_3 \), and probabilities \( \{p_1, p_2, p_3\} \) given by (assume \( x_1 \leq x_2 \leq x_3 \))
\[
a_1 = 1 + r_g, \quad p_1 = p(x), \quad a_2 = 1 + r + \sigma \varphi(x, \psi(x)), \quad p_2 = p(\varphi(x, \psi(x))), \quad a_3 = 1 + r + \sigma \psi(x), \quad p_3 = p(\psi(x)). \tag{5.10}
\]
Note that if \( x_2 < r_g \) then \( Y_k \) is a biatomic random variable, for which the results of Section 3 applies. To obtain the \( n = 3^T \) atoms of the discrete random variable \( V_T \), consider the set \( \Delta \) of all vectors \( \delta = (\delta_1, \ldots, \delta_T) \) with \( \delta_k \in \{-1,0,1\} \). Then each \( \delta \in \Delta \) yields an atom
\[
v_\delta = \sum_{k=1}^{T} c_k \prod_{j=k}^{T} \left[ a_1^{|\delta_1|} a_2^{|\delta_j|} a_3^{|\delta_j|} \right] \tag{5.11}
\]
with probability
\[
h_\delta = \Pr(V_T = v_\delta) = p_1^{|\delta_1|} p_2^{|\delta_2|} p_3^{|\delta_3|} \cdot \prod_{k=2}^{T} \left( (1 - \theta) \cdot p_1^{|\delta_k|} p_2^{|\delta_k|} p_3^{|\delta_k|} + \theta \cdot \epsilon(\delta_{k-1}, \delta_k) \right) \tag{5.12}
\]
where \( \varepsilon(x, y) = 1 \) if \( y = x \) and 0 else. Then proceed as at the end of Section 3.

Let us illustrate the obtained results at the guaranteed annuity-due with constant payments considered in Section 4 under the triatomic random walk return model with \( \theta = 0 \). As numerical example we use the data of Section 4 completed with the kurtosis parameter \( \gamma_2 = -0.8304 \). The triatomic return has support \( \left\{ x_1 = -0.328\%, x_2 = 4.493\%, x_3 = 8.395\% \right\} \). Its span \( x_3 - x_1 = 8.723\% \) is comparable to the observed span \( 10.71\% - 2.58\% = 8.13\% \). The triatomic returns \( Y_i - 1 \) have the support \( \left\{ a_1, a_2, a_3 \right\} = \{4.25\%, 4.493\%, 8.395\%\} \) and the probabilities \( \{p_1, p_2, p_3\} = \{0.01, 0.64011, 0.34989\} \). The cost of guarantee per unit of liability lies between 19 and 94 basic points below the prudent estimation of Section 4, and appears to us as a more “realistic” estimation. A dramatic improvement of the triatomic return model is felt in ERC and RAROC calculations. At the confidence level \( \alpha = 0.99 \) much less ERC is required. Our calculations are summarized in Table 5.1.

**Table 5.1**: ERC and RAROC of a guaranteed annuity-due

<table>
<thead>
<tr>
<th>( T )</th>
<th>( L_T )</th>
<th>( 100 \cdot C_T )</th>
<th>( 100 \cdot CVaR )</th>
<th>( \mu[V_T] )</th>
<th>( \sigma[V_T] )</th>
<th>( ICV[G_T] )</th>
<th>RAROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.043</td>
<td>0.56</td>
<td>0.56</td>
<td>1.059</td>
<td>0.019</td>
<td>0.549</td>
<td>1.755</td>
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<td>2.179</td>
<td>0.043</td>
<td>0.737</td>
<td>2.167</td>
</tr>
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<td>1.14</td>
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<td>3.365</td>
<td>0.075</td>
<td>0.880</td>
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<td>10.423</td>
<td>0.343</td>
<td>1.373</td>
<td>2.977</td>
</tr>
</tbody>
</table>
CHAPTER X. COMPOUND POISSON RISKS

1. CVaR bounds for compound Poisson risks.

An important Risk Management issue of an insurance company is the construction of more or less accurate bounds on risk measures like CVaR or ES for compound random sums \( S = X_1 + \ldots + X_N \), where the claim number \( N \) is Poisson \((\lambda)\), and the claim sizes \( X_i \) are independent and identically distributed as \( X \), and \( X_i \) is independent from \( N \). By incomplete information about the claim size, say \( X \) belongs to the set \( D = D([0,b],[\mu,\sigma]) \) of all non-negative random variables with maximum claim size \( b \), known mean \( \mu \) and standard deviation \( \sigma \), simple bounds are obtained as follows.

Following Hürlimann(1996a), Section 3, consider the stop-loss ordered extreme random variables \( \min X \) and \( \max X \) for the set \( D \), such that

\[
\min X \leq s_l X \leq s_u X \leq \max X, \quad \text{for all } X \in D. \tag{1.1}
\]

Then replace \( X \) by \( \min X \) and \( \max X \) in the compound Poisson random sums to get random sums \( S_{\min} \) and \( S_{\max} \) such that

\[
S_{\min} \leq s_l S \leq s_u S \leq S_{\max}, \quad \text{for all } X \in D. \tag{1.2}
\]

Since CVaR is preserved under stop-loss order by Proposition II.2.2, one obtains the bounds

\[
\text{CVaR}_\alpha[S_{\min}] \leq \text{CVaR}_\alpha[S] \leq \text{CVaR}_\alpha[S_{\max}] \quad \text{for all } \alpha \in [0,1]. \tag{1.3}
\]

For computational reasons, it is more advantageous to evaluate bounds based on finite atomic claim sizes. Now, the minimum \( X_{\min} \) is already 2-atomic while the maximum \( X_{\max} \) has a probability distribution of mixed discrete and continuous type. The latter can be replaced through mass dispersion by a 4-atomic stop-loss larger discrete approximation \( X_{\max} \leq s_l X_{\max}^d \) such that \( S_{\max} \leq s_l S_{\max}^d \) and \( \text{CVaR}_\alpha[S_{\max}] \leq \text{CVaR}_\alpha[S_{\max}^d] \) for all \( \alpha \in [0,1] \). Recall the structure of the finite atomic random variables \( X_{\min} \) and \( X_{\max}^d \). Let \( \nu = \left( \frac{\sigma}{\mu} \right)^2 \) the relative variance of the claim size, \( \nu_0 = \frac{b - \mu}{\mu} \) the maximum relative variance for the set \( D \) and \( \nu_r = \frac{\nu}{\nu_0} \) a relative variance ratio. The discrete supports and probabilities of these random variables are described for \( X_{\min} \) by

\[
\{x_1, x_2\} = \{(1 - \nu_r)\mu, (1 + \nu)\mu\}, \quad \{p_1, p_2\} = \left\{ \frac{\nu_0}{1 + \nu_0}, \frac{1}{1 + \nu_0} \right\}, \tag{1.4}
\]

and for \( X_{\max}^d \) by
\[ \{x_0, x_1, x_2, x_3\} = \left\{ 0, \frac{1}{2} (1 + v) \mu, \left[1 + \frac{1}{2} (v_0 - v) \right] \mu, (1 + v_0) \mu \right\}, \]
\[ \{p_0, p_1, p_2, p_3\} = \left\{ \frac{v}{1 + v}, \frac{v_0 - v}{(1 + v)(1 + v_0)}, \frac{v_0 - v}{(v_r + v_0)(1 + v_0)}, \frac{v_r}{v_r + v_0} \right\}. \quad (1.5) \]

In practice, one chooses the parameters and fixes a unit of money in such a way that the atoms \( x_i \) are non-negative integers. Recall that the probabilities \( f_k, k = 0, 1, 2, \ldots \) of a compound Poisson distribution with non-negative integer claim sizes \( x_0 = 0 < x_1 < \ldots < x_m \) and corresponding probabilities \( p_0, p_1, \ldots, p_m \) are best numerically evaluated using the Adelson-Panjer recursive algorithm (e.g. Panjer(1981), Hürlimann(1988a)):
\[
\begin{align*}
    f_0 &= e^{-\lambda(1 - p_0)}, \\
    f_k &= \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j) x_j p_j f_{k-x_j}, \quad k = 1, 2, 3, \ldots, \\
\end{align*} \quad (1.6)
\]
where \( \delta(x) = 1 \) if \( x \geq 0 \) and \( \delta(x) = 0 \) else. Finally, to obtain \( CVaR_{\alpha}[S] \) one uses the formulas (II.2.17) and (II.2.19).

Since computers only represent a finite number of digits, there remains to discuss the technical problems of round-off errors and underflow/overflow. Regarding round-off errors it has been shown by Panjer and Wang(1993) that the recursive formula (1.6) is strongly stable such that this algorithm works well. However, for large values of \( \lambda \) underflow/overflow occurs. In this situation, some methods have been proposed in Panjer and Willmot(1986). In Section 2, we use exponential scaling/de-scaling as follows. Let \( \mu_s = \mu \), \( \sigma_s^2 = \lambda (\mu^2 + \sigma^2) \) be the mean and variance of \( S \). Choose appropriately \( M = \mu_s - t \cdot \sigma_s \) for some \( t \) (\( t = 19, 25.5 \) in our example of Section 2 for \( \lambda = 2000, 3000 \)), and let \( r = \frac{\lambda(1 - p_0)}{M} \), \( m_0 = \lceil M \rceil \) the greatest integer less than \( M \). Exponential scaling and recursion yields
\[
\begin{align*}
    h_0 &= 1, \\
    h_k &= \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j) x_j p_j e^{-\lambda x_j} h_{k-x_j}, \quad k = 1, 2, \ldots, m_0. \\
\end{align*} \quad (1.7)
\]
Then apply exponential de-scaling setting \( f_k = h_k e^{r(k-M)} \), \( k = 0, \ldots, m_0 \), and continue the evaluation of \( f_k \) for \( k > m_0 \) with the recursion (1.6).

2. ERC bounds and normal approximation for an insurance risk business.

We are interested in the evaluation of economic risk capital of an insurance portfolio whose compound Poisson aggregate claims \( S \) at a future date are covered by a risk premium \( P > \mu_s \). The future random loss of the portfolio can be decomposed as follows:
\[
S - P = (\mu_s - P) + (S - \mu_s). \quad (2.1)
\]
The first component, which is the negative of the insurance margin, represents the future expected insurance gain and belongs to the stakeholders of the insurance company. To protect this expected gain, one requires some economic risk capital to cover the insurance loss \( L = S - \mu_s \) (signed deviation from the mean aggregate claims). Using CVaR as risk measure, the future value of this economic risk capital is equal to

\[
CVaR_\alpha[L] = CVaR_\alpha[S] - \mu_s, \tag{2.2}
\]

where \( \alpha \) is some prescribed confidence level. Note that the equality in (2.2) follows from the translation invariant property of CVaR, which is one of the axioms required to define a coherent risk measure.

The following numerical illustration is based on the approximate figures of a real-life portfolio of grouped life insurance contracts from the early 1980’s. For some unit of money, our choice for the claim size parameters is \( \mu = 12, \sigma^2 = 360, b = 48 \), hence \( \nu = \frac{5}{2}, v_0 = 3, v_r = \frac{5}{6} \). According to (1.4) and (1.5) the discrete supports and probabilities are given for \( X_{\text{min}} \) by

\[
\{x_1, x_2\} = \{2.42\}, \quad \{p_1, p_2\} = \{0.75,0.25\}, \tag{2.3}
\]

and for \( X_{\text{max}} \) by

\[
\{x_0, x_1, x_2, x_3\} = \{0.21,25,48\}, \quad \{p_0, p_1, p_2, p_3\} = \{0.71429,0.03571,0.03261,0.21739\}. \tag{2.4}
\]

Table 2.1 displays the values \( CVaR_{\text{min}} = \frac{1}{\mu_s} \cdot CVaR_\alpha[L_{\text{min}}] \) and \( CVaR_{\text{max}} = \frac{1}{\mu_s} \cdot CVaR_\alpha[L^{d}_{\text{max}}] \) with \( L_{\text{min}} = S_{\text{min}} - \mu_s \), \( L^{d}_{\text{max}} = S^{d}_{\text{max}} - \mu_s \), which represent bounds on the insurance economic risk capital per unit of mean aggregate claims for \( \alpha = 95\%, 99\%, 99.75\% \) by varying the expected number of claims \( \lambda \). The average rate

\[
CVaR^A = \frac{1}{2} \left( CVaR_{\text{min}} + CVaR_{\text{max}} \right) \tag{2.5}
\]

is compared with the normal approximation rate

\[
CVaR^N = \frac{1}{\epsilon} \phi\left[\epsilon^{-1}(\alpha)\right] \cdot \frac{\sigma_s}{\mu_s}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \phi(x) = \Phi'(x), \tag{2.6}
\]

which is obtained approximating \( S \) by a normal random variable \( S^N \) with mean \( \mu_s \) and standard deviation \( \sigma_s \). The approximation error is measured here by the signed normal deviation rate

\[
D^N = CVaR^N - CVaR^A. \tag{2.7}
\]

The following observations are noted. By fixed confidence level \( \alpha \), the normal approximation underestimates the average rate up to some fixed rather large expected number...
of claims $\lambda$, and then overestimates it. The underestimation increases by increasing confidence level $\alpha$. Since computational difficulties with the exponential scaling/de-scaling method of Section 1 arise for values of $\lambda$ beyond 3000, the normal approximation appears useful in this range provided insurers agree to set insurance economic risk capital rates at the proposed average rate (2.5).

**Table 2.1**: CVaR bounds and normal approximation as percentages of $\mu_s$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\lambda$</th>
<th>$CVaR_{min}$</th>
<th>$CVaR_{max}$</th>
<th>$CVaR_\lambda$</th>
<th>$CVaR_N$</th>
<th>$D_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>95%</td>
<td>100</td>
<td>38.123</td>
<td>41.944</td>
<td>40.033</td>
<td>38.590</td>
<td>−1.443</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>26.571</td>
<td>29.232</td>
<td>27.901</td>
<td>27.287</td>
<td>−0.614</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>21.554</td>
<td>23.711</td>
<td>22.632</td>
<td>22.280</td>
<td>−0.352</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>18.593</td>
<td>20.453</td>
<td>19.523</td>
<td>19.295</td>
<td>−0.228</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>16.585</td>
<td>18.244</td>
<td>17.414</td>
<td>17.258</td>
<td>−0.156</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>11.648</td>
<td>12.812</td>
<td>12.230</td>
<td>12.203</td>
<td>−0.027</td>
</tr>
<tr>
<td></td>
<td>2000</td>
<td>8.197</td>
<td>9.015</td>
<td>8.606</td>
<td>8.629</td>
<td>0.023</td>
</tr>
<tr>
<td></td>
<td>3000</td>
<td>6.678</td>
<td>7.345</td>
<td>7.011</td>
<td>7.046</td>
<td>0.035</td>
</tr>
<tr>
<td>99%</td>
<td>100</td>
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<td>55.297</td>
<td>52.774</td>
<td>49.862</td>
<td>−2.912</td>
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<td>−0.564</td>
</tr>
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<td>15.912</td>
<td>15.768</td>
<td>−0.144</td>
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<tr>
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<td>2000</td>
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<td>11.174</td>
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</tr>
<tr>
<td></td>
<td>3000</td>
<td>8.663</td>
<td>9.529</td>
<td>9.096</td>
<td>9.103</td>
<td>0.007</td>
</tr>
<tr>
<td>99.75%</td>
<td>100</td>
<td>59.333</td>
<td>65.315</td>
<td>62.342</td>
<td>58.077</td>
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<td>40.987</td>
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<td>43.045</td>
<td>41.067</td>
<td>−1.978</td>
</tr>
<tr>
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<td>33.109</td>
<td>36.430</td>
<td>34.770</td>
<td>33.531</td>
<td>−1.239</td>
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<td>31.343</td>
<td>29.916</td>
<td>29.039</td>
<td>−0.877</td>
</tr>
<tr>
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<td>27.908</td>
<td>26.637</td>
<td>25.973</td>
<td>−0.664</td>
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<tr>
<td></td>
<td>1000</td>
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<td>19.510</td>
<td>18.622</td>
<td>18.366</td>
<td>−0.256</td>
</tr>
<tr>
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<td>3000</td>
<td>10.119</td>
<td>11.130</td>
<td>10.625</td>
<td>10.603</td>
<td>−0.022</td>
</tr>
</tbody>
</table>
CHAPTER XI. LIFE INSURANCE BUSINESS

1. Introduction.

A major problem faced by insurance companies is the determination of capital requirement and the associated cost of capital for the insurance and investment risks of portfolios of insurance policies. To evaluate these quantities for a life insurance company, we consider a simple stochastic model of a life portfolio in a multi-period setting, which allows one to determine its accumulated aggregate loss over a finite number of insurance periods. This is the main ingredient required for the evaluation of an appropriate capital requirement. Details are found in Section 2.

Then, in Section 3, we follow the recent approach by Dhaene and Goovaerts (2002) (see also Chapter V and VI), which propose a method for deriving optimal amounts of economic risk capital and cost of capital. The basic idea consists to minimize the cost of capital function defined as the sum of the interest on capital and the cost of solvability. The latter is the theoretical price associated to the insolvency risk, that is the market price of a stop-loss contract with the end of period value of the economic risk capital as deductible. We assume that market prices are set using a distortion measure as justified on an axiomatic basis by Wang et al. (1997). The obtained optimal economic risk capital identifies with the value-at-risk at some specific confidence level of the accumulated aggregate loss of the life portfolio discounted at the risk-free rate. Similarly, the end of period value of the minimum cost of capital identifies with the interest differential on the distorted conditional value-at-risk of the accumulated aggregate loss taken at the borrowing and risk-free interest rates.

Section 4 illustrates the impact of the considered method for life insurance companies. Based on several justifying arguments, the accumulated aggregate loss of a life portfolio is approximated by a normal distribution. In this situation, one gets simple explicit formulas for the optimal economic risk capital and cost of capital, which can be evaluated using spreadsheet calculations. We evaluate these formulas and their sensitivity with respect to two essential parameters, namely the time horizon and the borrowing interest rate on the economic risk capital. Fixing the other parameters, we retain only those feasible numerical solutions for which the confidence level is at least 95% and the time horizon at most 5 years. It is interesting to discuss the obtained numerical figures. One observes that the required optimal economic capital is minimum for the longest five year time horizon. The optimal cost of capital is decreasing with the borrowing interest rate and increases with the time horizon by fixed borrowing interest rate. The dependence of the optimal cost of capital on the confidence level is more stable but decreases with increasing confidence level. The ultimate choice will depend on further legal and market restrictions about the involved parameters.

2. The aggregate loss of a life insurance portfolio.

We consider a simple stochastic model of a life portfolio in a multi-period setting, which allows one to determine its accumulated aggregate loss over a finite number of insurance periods. This quantity is essential for a life insurance company because it is the main ingredient required to evaluate the economic risk capital and the cost of capital associated to life insurance portfolios.

Our simplified insurance business model takes into account the insurance risk associated to random insurance claims and the investment risk associated to random returns on invested capital. It is assumed throughout that the random insurance claims and the random returns are
stochastically independent. The management risk associated to administration expenses and the attitude to risk are not taken into account here. The technical values of a traditional life portfolio can be summarized as follows (e.g. Bowers et al. (1986), p. 209):

\[
\pi_t^R : \text{the risk premiums in period } [t-1,t] \text{ due at time } t-1
\]
\[
\pi_t^S : \text{the saving premiums in period } [t-1,t] \text{ due at time } t-1
\]
\[
\pi_t = \pi_t^R + \pi_t^S : \text{the net premiums}
\]
\[
V : \text{the actuarial reserves required at time } t
\]
\[
S_t : \text{the random aggregate claims in period } [t-1,t] \text{ due at time } t
\]
\[
I_t : \text{the random return on investment in period } [t-1,t]
\]
\[
i = r = 1 + r : \text{the one-period constant technical interest rate}
\]
\[
v = r^{-1} : \text{the technical discount rate}
\]

Recall some well-known relationships. The risk premiums are equal to the sum of the discounted value of the expected aggregate claims, denoted \(E[S_t]\), and the loading on aggregate claims, denoted \(l[S_t]\). In formulas one has

\[
\pi_t^R = v \cdot (E[S_t] + l[S_t]). \tag{2.1}
\]

The discounted reserves at time \(t\) consists of the reserves at time \(t-1\) and the saving premiums, that is

\[
v \cdot V = \pi_t^S. \tag{2.2}
\]

One is interested in the aggregate loss of the life portfolio at a future time \(T\), that is after a number \(T\) of insurance periods. The random value of the liability \(L_T\) at time \(T\) of the portfolio consists of the actuarial reserves at time \(T\) and the random accumulated value of the aggregate claims in each period \([t-1,t], \ t = 1, \ldots, T\), that is

\[
L_T = v \cdot V + \sum_{t=1}^{T} r^{T-t} \cdot S_t. \tag{2.3}
\]

Discounted at the technical interest rate, the present value of the assets \(A_0\) at time \(t = 0\) of the life portfolio consists of the actuarial reserves and the discounted value of all future premiums, that is

\[
A_0 = v \cdot V + \sum_{t=1}^{T} v^{t-1} \cdot \pi_t. \tag{2.4}
\]

The aggregate loss at time \(T\), defined as difference between liabilities and assets, is denoted and equal to \(X_T = L_T - A_0 \cdot R_T\), where \(R_T = \prod_{t=1}^{T} (1+I_t)\) represents the random accumulated return on investment over the period \([0,T]\). Using (2.2) one obtains through induction the actuarial reserves at time \(T\).
\[ rV = r^{T_0} V + \sum_{i=1}^{T} r^{T-i+1} \cdot \pi_i^T. \]  

(2.5)

Using (2.1)-(2.5) the aggregate loss can be rewritten as

\[ X_T = A_0 \cdot (r^T - R_T) + \sum_{i=0}^{T-1} r^{T-i-1}(S_{T+i} - E[S_{T+i}]) - \sum_{j=0}^{T-1} r^{T-j-1} \cdot E[S_{T+j}]. \]  

(2.6)

The first term, abbreviated \( X_T^{\text{inv}} \), represents the possible random investment loss, which is positive in case the technical interest cannot be guaranteed and is called investment risk. The second term represents the possible random insurance loss as deviation of the random accumulated value of the aggregate claims in each period from its expected value and is called insurance risk, abbreviated \( X_T^{\text{ins}} \). Finally, the third term represents the expected accumulated insurance surplus at time \( T \) provided by the risk premium margins and is called insurance margin, abbreviated \( M_T^{\text{ins}} \). For (2.6) one uses the following short hand notation

\[ X_T = X_T^{\text{inv}} + X_T^{\text{ins}} - M_T^{\text{ins}}. \]  

(2.7)

In practice, it is necessary to specify a model of the accumulated aggregate claims (e.g. Hürlimann(2002c) for an individual multi-period life model with one cause of decrement) and a distribution for the random accumulated return on investment (e.g. a normal, elliptical, log-normal distribution, etc.).

3. **An optimal method.**

A major problem faced by insurance companies is the determination of capital requirement and the associated cost of capital for the insurance and investment risks of portfolios of insurance policies. A basic approach consists to apply an appropriate risk measure to the loss and profit distribution, which takes into account its shape, especially its right tail. Standard choices nowadays are the value-at-risk measure (VaR) and the conditional value-at-risk measure (CVaR) or equivalently the expected shortfall measure (ES) (see Section II.2). In the present Chapter, we follow the approach by Dhaene and Goovaerts(2002), which seeks after an optimal method.

Recall that the random variable \( X_T \) in (2.7) represents the aggregate loss of an insurance life portfolio at the end of a multi-period \([0, T]\). To avoid the insolvency risk, an insurer borrows at the beginning of the period and at the interest rate \( i_c \) some economic risk capital \( C = ERC[X_T] \), which is assumed to depend only on the random loss. The insurance company invests this capital at the risk-free interest rate \( i_f \). In the following, denote by \( r_c = 1 + i_c \) and \( r_f = 1 + i_f \) the one-period accumulated interest rates corresponding to \( i_c, i_f \).

It is natural to expect that the end of period resulting (net) interest on capital \( \left[ r_c^T - r_f^T \right] \cdot C \) should be as small as possible. On the other hand, insolvency occurs if \( X_T > C \cdot r_f^T \), hence \( C \) should be as large as possible. Therefore, an “optimal” compromise solution must be found. The theoretical price, required to eliminate the insolvency risk, is the market price of a stop-loss contract with payoff \( X_T - C \cdot r_f^T \). If the price of such a contract is set using a risk
measure \( R[\cdot] \), then the cost of solvability equals \( R[\{X_T - C \cdot r_f^T \}] \). The aggregate cost of solvability and interest on capital determines the cost of capital function \( f(C) = CoC[X_T] = R[\{X_T - C \cdot r_f^T \}] + [r_c^T - r_f^T] \cdot C \), which should be minimized. As in Section V.2, we assume that market prices are set using a distortion measure such that

\[
R[X] = \int_{-\infty}^{\infty} F_X^g(x) \, dx - \int_{-\infty}^{0} [1 - F_X^g(x)] \, dx, \quad (3.1)
\]

for all random variables \( X \) from a given set of financial losses, where \( F_X^g(x) = g(F_X(x)) \) denotes the distorted survival function associated to the survival function \( F_X(x) = 1 - F_X(x) \), \( F_X(x) = P(X \leq x) \) the distribution function of \( X \), and \( g : [0,1] \to [0,1] \) is a continuous increasing function such that \( g(0) = 0 \) and \( g(1) = 1 \), called distortion function. Recall that the distortion measure (3.1) is a coherent risk measure provided the distortion measure \( x g \) is a concave function. It is interpreted as expectation of \( X \) with respect to the distorted survival function and is denoted in the following by \( R[X] = E^g[X] \).

With this choice of risk measure, the cost of capital function can be rewritten as

\[
f(C) = E^g[\{X_T - C \cdot r_f^T \}] + [r_c^T - r_f^T] \cdot C, \quad (3.2)
\]

Under the assumption of continuous distributions, the optimal economic risk capital, which minimizes (3.1), and the corresponding minimum cost of capital are determined as follows (formulas (5) and (7) in Dhaene and Goovaerts(2002)):

\[
C = ERC[X_T] = \frac{1}{r_f^T} \cdot \left( F_X^g \right)^{-1} \left( \frac{r_c^T - r_f^T}{r_f^T} \right) = \frac{1}{r_f^T} \cdot F_X^g \left( 1 - g^{-1} \left( \frac{r_c^T - r_f^T}{r_f^T} \right) \right), \quad (3.3)
\]

\[
f(C) = \frac{r_c^T - r_f^T}{r_f^T} \cdot E^g[\{X_T \mid X_T > r_f^T \} \cdot C] \quad (3.4)
\]

The formula (3.3) identifies the end of the period value of the optimal economic risk capital, that is \( r_f^T \cdot ERC[X_T] \), as the value-at-risk of \( X_T \) at the confidence level \( \alpha = 1 - g^{-1} \left( \frac{r_c^T - r_f^T}{r_f^T} \right) \), that is \( r_f^T \cdot ERC[X_T] = VaR_\alpha[X_T] \) in the usual notation. Similarly, the end of the period value of the minimum cost of capital identifies with the interest differential at the rates \( i_c, i_f \) on the distorted conditional value-at-risk of \( X_T \) at the same confidence level evaluated with respect to the distorted survival function, that is

\[
CoC[X_T] = \frac{r_c^T - r_f^T}{r_f^T} \cdot E^g[\{X_T \mid X_T > VaR_\alpha[X_T] \}] = \frac{r_c^T - r_f^T}{r_f^T} \cdot CVaR_\alpha[X_T],
\]

where the latter notation remembers the usual notation of conditional value-at-risk for continuous distributions, except for the upper index \( g \). In this setting, the considerable amount of recent research related to these risk measures is open for application.
4. **Normal approximation of the aggregate loss.**

The optimal economic risk capital and cost of capital formulas (3.3) and (3.4) depend on the distortion function \( g(x) \), for which a best choice remains to be determined. Chapter V attempts to justify the choices \( g(x) = \sqrt{x} \) and \( g(x) = x \) on an axiomatic basis. For simplicity, we restrict here our attention to the trivial choice \( g(x) = x \). In this situation, the cost of solvability identifies with the expected net cost of the required stop-loss contract without any extra loading needed to absorb the fluctuations occurring in realizations of the stop-loss payoff. The corresponding optimal economic risk capital and cost of capital can be viewed as an absolute minimum, which could be tolerated by the regulator.

To illustrate the impact of this method for life insurance, we assume that the aggregate claims \( S_t, \ t = 1, \ldots, T \), are independent and identically distributed, where the independence assumption can be justified as in Hürlimann(2002c), Section 4, at least for the cost of capital, which preserves the stop-loss order. It follows that the mean and standard deviation of the insurance risk are given by

\[
\mu_{ins}^T = E[X_{ins}^T] = 0, \quad \sigma_{ins}^T = \sqrt{Var[X_{ins}^T]} = \sqrt{\frac{1 - r^{2T}}{1 - r^2} \cdot \sigma_S}, \quad \sigma_S^2 = Var[S]. \tag{4.1}
\]

We assume further that the risk premiums \( \pi_t^R, \ t = 1, \ldots, T \), are constant, so that the insurance margin reads

\[
M_{ins}^T = \frac{1 - r^T}{1 - r} \cdot r \pi_t^R, \tag{4.2}
\]

where \( r \pi_t^R - E[S_t] \) is the one-period insurance risk margin, expressed in units of the end of year risk premium. Further, let \( \mu_R^T \) and \( \sigma_R \sqrt{T} \) be the mean and standard deviation of the accumulated return on investment \( R_t \), where \( \mu_R \) and \( \sigma_R \) are interpreted as a constant mean and standard deviation of the one-period accumulated return on investment. With these definitions, the mean and standard deviation of the investment risk are given by

\[
\mu_{inv}^T = E[X_{inv}^T] = (r^T - \mu_R^T) \cdot A_0, \quad \sigma_{inv}^T = \sqrt{Var[X_{inv}^T]} = \sigma_R \sqrt{T} \cdot A_0. \tag{4.3}
\]

For numerical evaluations, it is necessary to specify distributions for the insurance risk \( X_{ins}^T \), the investment risk \( X_{inv}^T \) as well as a bivariate distribution for the random couple \( (X_{inv}^T, X_{ins}^T) \). For practical simplicity, we will assume that \( X_{inv}^T \) can be approximated by a normal distribution and that the random variables \( X_{inv}^T \) and \( X_{ins}^T \) are independent. As argued in Appendix A, it is possible to approximate the distribution of \( S_t \) by a gamma distribution. By Hürlimann(2001a), Example 4.2, the independent sum \( \sum_{t=1}^{T} r^{T-t} \cdot S_t \) produces VaR and CVaR values, which are very close to those of a gamma distributed sum. Therefore, in calculations, the distribution of \( X_{ins}^T \) can be well approximated by a translated gamma distribution. This method could be followed for comparative purposes. Another approach, advocated in Section X.2, consists to construct using stochastic orders lower and upper
bounds \( X_{T,\text{min}}^{\text{ins}} \) and \( X_{T,\text{max}}^{\text{ins}} \) on \( X_T^{\text{ins}} \), calculate corresponding VaR and CVaR extremal values, and use their average value as estimation. At least for CVaR, it turns out that the obtained average value is close to the value obtained from a normal approximation to \( X_T^{\text{ins}} \). In some cases, there is also another justification for a normal approximation to \( X_T^{\text{ins}} \). When the present value of the assets \( A_0 \) is very big compared to the contribution from the insurance risk, as observed in some life insurance companies, the random term \( X_T^{\text{ins}} \) will only disturb slightly the resulting sum \( X_T \), and the latter behaves like \( X_T^{\text{inv}} \), that is like a normally distributed random variable. Under the assumption of a normally distributed aggregate loss \( X_T \), the optimal economic risk capital and cost of capital of Section 3 are determined by the following explicit formulas:

\[
\text{ERC}[X_T] = \frac{1}{r_f} \cdot \text{VaR}_a[X_T] = \frac{1}{r_f} \left( r^T - \mu^T_R + \Phi^{-1}(\alpha) \cdot \sqrt{\sigma^2_R T + \left( \frac{\sigma^2_{\text{inv}}}{\lambda_0} \right)^2} \cdot A_0 - M^{\text{ins}}_T \right)
\]

(4.4)

\[
\text{CoC}[X_T] = \frac{r^C - r^T}{r_f} \cdot \text{CVaR}_a[X_T] = \frac{r^C - r^T}{r_f} \left( r^T - \mu^T_R + \frac{1}{1-\alpha} \varphi\left[\Phi^{-1}(\alpha)\right] \cdot \sqrt{\sigma^2_R T + \left( \frac{\sigma^2_{\text{inv}}}{\lambda_0} \right)^2} \cdot A_0 - M^{\text{ins}}_T \right)
\]

(4.5)

\[
\alpha = 1 - \left( \frac{r^C - r^T}{r_f} \right)
\]

(4.6)

In these formulas \( \Phi^{-1}(\alpha) \) denotes the \( \alpha \)-quantile of the standard normal distribution, and \( \varphi(x) \) is the standard normal density.

It appears useful to illustrate these formulas and their sensitivity with respect to two essential parameters, namely the time horizon and the borrowing interest rate on the economic risk capital. Fixing the other parameters, we retain only those feasible numerical solutions for which the confidence level is at least 95\% and the time horizon at most 5 years. Our example in Table 4.1 is based on the following fixed portfolio characteristics:

\[
\begin{align*}
A_0 &= 1'000'000 \\
\pi^R &= 1'000 \\
\nu &= 0.3 \\
\sigma^S &= 1435 \\
r &= 1.025 \\
r_f &= 1.0325 \\
\mu_R &= 1.05 \\
\sigma_R &= 0.05
\end{align*}
\]
**Table 4.1**: confidence level percentages, cost of capital basis points (100 bp = 1%), and economic risk capital percentages

<table>
<thead>
<tr>
<th>$i_c$</th>
<th>4%</th>
<th>4.5%</th>
<th>5%</th>
<th>5.5%</th>
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<th>7%</th>
<th>7.5%</th>
<th>8%</th>
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<tr>
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<tr>
<td>CoC</td>
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<td>16</td>
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<td>23</td>
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<tr>
<td>ERC</td>
<td>9.12</td>
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<td>CoC</td>
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<tr>
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<tr>
<td>$\alpha$</td>
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<td>95.1</td>
<td>95.1</td>
<td>95.1</td>
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<td>95.1</td>
<td>95.1</td>
<td>95.1</td>
<td>95.1</td>
</tr>
<tr>
<td>CoC</td>
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<td>31</td>
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<td>31</td>
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<td>ERC</td>
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<td></td>
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</tr>
</tbody>
</table>

It is interesting to discuss the obtained numerical figures. One observes that the required optimal economic capital is minimum for the longest five year time horizon. The optimal cost of capital is decreasing with the borrowing interest rate and increases with the time horizon by fixed borrowing interest rate. The dependence of the optimal cost of capital on the confidence level is more stable but decreases with increasing confidence level. The ultimate choice will depend on further legal and market restrictions about the involved parameters.
CHAPTER XII. UNIT-LINKED LIFE PORTFOLIOS

1. **Introduction.**

The present Chapter shows how the multi-period economic risk capital of a life portfolio can be evaluated. Defining the economic risk capital as the value-at-risk or conditional value-at-risk associated to the multi-period aggregate surplus of a life portfolio, the last quantity must be first determined. For this, the multi-period aggregate surplus is divided into three components, namely the investment risk, the insurance risk and the insurance margin, where each component is modeled separately. Our application concerns the evaluation of the economic risk capital of a portfolio of unit-linked endowment policies with guarantee and its value based comparison with a portfolio of traditional endowment policies. Using a simple but reliable gamma approximation of the insurance risk component, it is shown that the guaranteed unit-linked contract performs better than the traditional one on a risk-adjusted return scale. Indeed, the risk-adjusted return on capital, abbreviated RAROC, that is the expected gain per unit of economic risk capital is greater for the guaranteed unit-linked product than for the traditional one. A more detailed outline follows.

Section 2 surveys the analysis and pricing of unit-linked contracts with guarantees, whose main feature is the inclusion of random benefits as opposed to deterministic benefits in traditional life insurance. Inspired by the developments in Aase and Persson(1994), we offer an analytical treatment close to the traditional viewpoint, which can be easily applied to the evaluation of economic risk capital along the presented approach.

Section 3 illustrates the actuarial use of our results. In Section 3.1 we show by example that the distribution of the random multi-period aggregate loss of a life portfolio can be well approximated using a gamma distribution assumption for the insurance risk. Making this approximation, we compare in Section 3.2 the traditional endowment insurance with the guaranteed unit-linked endowment insurance and obtain an improved risk-adjusted performance for the latter.

2. **Analysis and pricing of unit-linked life insurance with guarantee.**

The analysis and pricing of unit-linked contracts with guarantees, whose main feature is the inclusion of random benefits as opposed to deterministic benefits in traditional life insurance, has been discussed in the literature since about 25 years by many researchers, in particular Brennan and Schwartz(1976/79a/b), Boyle and Schwartz(1977), Corby(1977), Delbaen(1986), Delvaux and Magnée(1991), Bacinello and Ortu(1993), Aase and Persson(1994), Nielsen and Sandmann(1995), Kurz(1996).

Inspired by the developments in Aase and Persson(1994), we offer an analytical treatment of unit-linked contracts with guarantee, which is very close to the traditional viewpoint, and thus can be easily applied to the evaluation of economic risk capital. Our attention focuses on combinations of pure endowment and term insurance contracts, especially unit-linked endowment contracts with guarantee against constant periodic premiums. For clearness the presentation is divided into four Subsections.
2.1. Unit-linked contracts.

The description is based on the following quantities:

\( x \): age at entry of a policy-holder
\( s \): age at expiration of a contract
\( T = s - x \): term of a contract
\( t \): current time of valuation
\( G(t) \): guaranteed death benefit at time \( t < T \)
\( G(T) \): guaranteed benefit at expiration of a contract
\( v \): technical discount rate
\( S(t) \): random market value at time \( t \) of one share of the unit-linked reference portfolio
\( N(t) \): prescribed number of shares at time \( t \) of the reference portfolio included in the benefit
\( j \): expected return of the reference portfolio over the time horizon \([0, T]\)
\( \sigma \): volatility of the reference portfolio

One observes that the guaranteed benefit at time \( t \) is covered through shares of the reference portfolio provided one has

\[
N(t)S(t) = G(t).
\] (2.1)

However, at time \( t = 0 \) of underwriting the future value \( S(t) \) of the reference portfolio is unknown. Therefore, for the purpose of pricing, we assume that the expected value of the shares in the reference portfolio covers the guaranteed benefit, that is \( N(t)E[S(t)] = G(t) \). It follows that the prescribed number of shares is determined by

\[
N(t) = G(t)w', \quad w = (1 + j)^{-1}.
\] (2.2)

If the market value of the shares in the reference portfolio exceeds the guaranteed benefit at death or expiration, then the difference is paid as bonus to the beneficiary of the contract. The financial payoff at time \( t \) satisfies the relationship

\[
\max\{N(t)S(t), G(t)\} = G(t) + (N(t)S(t) - G(t))_+.
\] (2.3)

With (2.3) the payoff is decomposed in a deterministic payment \( G(t) \) and a stochastic payment from a call-option on the market value of the shares with exercise price \( G(t) \).

2.2. Pricing of the unit-linked endowment contract with guarantee.

The benefit decomposition (2.3) suggests to represent technical values of unit-linked contracts as sums of technical values of traditional contracts with deterministic payoff \( G(t) \) and of technical values of contracts with stochastic call-option payoff \( (N(t)S(t) - G(t))_+ \). The endowment insurance is the superposition of a term insurance and a pure endowment insurance. Therefore, each technical value can be additionally decomposed in a term and pure
endowment component. Following this procedure, the two-fold decomposition of the relevant technical values with their notations are summarized in the Tables 2.1 and 2.2.

### Table 2.1: Decomposition of single premiums

<table>
<thead>
<tr>
<th>benefit</th>
<th>insurance</th>
<th>term insurance</th>
<th>pure endowment</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>traditional</td>
<td>$TI_x$</td>
<td>$TPE_x$</td>
<td>$TE_x$</td>
<td></td>
</tr>
<tr>
<td>unit-linked call-option</td>
<td>$TIC_x$</td>
<td>$TPEC_x$</td>
<td>$TEC_x$</td>
<td></td>
</tr>
<tr>
<td>guaranteed unit-linked</td>
<td>$TIG_x$</td>
<td>$TPEG_x$</td>
<td>$TEG_x$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2.2: Decomposition of actuarial reserves

<table>
<thead>
<tr>
<th>benefit</th>
<th>insurance</th>
<th>term insurance</th>
<th>pure endowment</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>traditional</td>
<td>$V^TI_x$</td>
<td>$V^PE_x$</td>
<td>$V^E_x$</td>
<td></td>
</tr>
<tr>
<td>unit-linked call-option</td>
<td>$V^{TIC}_x$</td>
<td>$V^{PEC}_x$</td>
<td>$V^{EC}_x$</td>
<td></td>
</tr>
<tr>
<td>guaranteed unit-linked</td>
<td>$V^{TIG}_x$</td>
<td>$V^{PEG}_x$</td>
<td>$V^{EG}_x$</td>
<td></td>
</tr>
</tbody>
</table>

The remaining required technical values for each component are derived from the single premium, written in general notation as $E_x$, the actuarial reserves $V_x$, $V_{x-1}$, and the discount rate $v$ using the following formulas:

$$\pi_s = \frac{E_x}{d_{x:T}}$$ : periodic net premium \hfill (2.4)

$$\pi^S_s = v \cdot V_x - v \cdot V_{x-1}$$ : saving premium in period $[t-1,t]$ \hfill (2.5)

$$\pi^R_s = \pi^R_s - \pi^S_s$$ : risk premium in period $[t-1,t]$ \hfill (2.6)

$$S_s = \pi^R_s \cdot (vq_{x+1})^{-1}$$ : sum at risk in period $[t-1,t]$ \hfill (2.7)

### 2.3. Analytical formulas.

The valuation of the traditional insurance contracts with deterministic payoff is classical and well-known. Present values of the unit-linked call-option contracts are determined using Black-Scholes formula, where the riskless rate is set equal to the technical interest rate. It follows that the value of the call-option at time $t$ with payoff $(N(u)S(u) - G(u))$, at time $u > t$, abbreviated $C(t,u)$, is given by (use the relationship (2.2)):

$$C(t,u) = N(u)S(t)\Phi[d_1(t,u)] - G(u)v^{u-t}\Phi[d_1(t,u)]$$ \hfill (2.8)

$$d_1(t,u) = \frac{\ln\left(\frac{N(u)S(t)}{G(u)}\right) + \left(\delta + \frac{1}{2}\sigma^2\right)(u-t)}{\sigma\sqrt{u-t}},$$
\[ d_z(t,u) = d_i(t,u) - \sigma \sqrt{u - t}, \quad \delta = -\ln(v). \]

At the time \( t = 0 \) of underwriting, the future value \( S(t) \) is unknown. As in Section 2.1 one assumes in formula (2.8) that the future value at time \( t \) of a share of the reference portfolio coincides with its expected value, that is \( S(t) = (1 + j)^t \). Under this assumption, the required technical values are calculated explicitly as follows:

**Single premiums**

\[ TI_{x\tau} = \sum_{k=0}^{T-1} G(k+1) \cdot p_x \cdot q_{x+k} \]  
(2.9)

\[ PE_x = p_x \cdot v^T \cdot G(T) \]  
(2.10)

\[ TIC_{x\tau} = \sum_{k=0}^{T-1} C(0,k+1) \cdot p_x \cdot q_{x+k} \]  
(2.11)

\[ PEC_x = p_x \cdot C(0,T) \]  
(2.12)

\[ E_x = TI_{x\tau} + PE_x \]  
(2.13)

\[ EC_x = TIC_{x\tau} + PEC_x \]  
(2.14)

\[ TIG_{x\tau} = TI_{x\tau} + TIC_{x\tau} \]  
(2.15)

\[ PEG_x = PE_x + PEC_x \]  
(2.16)

\[ EG_x = E_x + EC_x = TIG_{x\tau} + PEG_x \]  
(2.17)

**Actuarial reserves**

\[ V^T_{x\tau} = \sum_{k=t}^{T-1} G(k+1) \cdot v^{k+1-t} \cdot k \cdot p_{x+t} \cdot q_{x+k} - \sum_{k=t}^{T-1} \pi^T_{x} \cdot v^{k-t} \cdot k \cdot p_{x+t} \cdot q_{x+k} \]  
(2.18)

\[ V^P_{x\tau} = p_x \cdot v^T \cdot G(T) - \sum_{k=t}^{T-1} \pi^P_{x} \cdot v^{k-t} \cdot k \cdot p_{x+t} \cdot q_{x+k} \]  
(2.19)

\[ V^{TIC}_{x\tau} = \sum_{k=t}^{T-1} C(t,k+1) \cdot (k-t) \cdot p_{x+t} \cdot q_{x+k} - \sum_{k=t}^{T-1} \pi^{TIC}_{x} \cdot v^{k-t} \cdot (k-t) \cdot p_{x+t} \cdot q_{x+k} \]  
(2.20)

\[ V^{PEC}_{x\tau} = p_x \cdot C(t,T) - \sum_{k=t}^{T-1} \pi^{PEC}_{x} \cdot v^{k-t} \cdot (k-t) \cdot p_{x+t} \cdot q_{x+k} \]  
(2.21)

\[ V^E_{x\tau} = V^T_{x\tau} + V^P_{x\tau} \]  
(2.22)

\[ V^{EC}_{x\tau} = V^{TIC}_{x\tau} + V^{PEC}_{x\tau} \]  
(2.23)

\[ V^{TIG}_{x\tau} = V^T_{x\tau} + V^{TIC}_{x\tau} \]  
(2.24)

\[ V^{PEG}_{x\tau} = V^P_{x\tau} + V^{PEC}_{x\tau} \]  
(2.25)

\[ V^{EG}_{x\tau} = V^E_{x\tau} + V^{EG}_{x\tau} = V^{TIG}_{x\tau} + V^{PEG}_{x\tau} \]  
(2.26)
2.4. A numerical illustration.

To illustrate numerically the obtained formulas, consider a unit-linked endowment policy with guaranteed payment \( G(t) = 100, t = 1, \ldots, T \), for a man aged \( x = 50 \) at entry with a term of \( T = 10 \) years. To calculate single premiums and actuarial reserves we use second order probabilities of death (here 60% of the Swiss Life Table GKM80) and a technical interest rate \( i = 2.5\% \). The expected return and the volatility of the reference portfolio are assumed to take the values \( j = 7\% \) and \( \sigma = 15\% \). Numerical results for the various quantities of interest are summarized in the Tables below.

**Table 2.3**: Single premiums

<table>
<thead>
<tr>
<th>Product</th>
<th>Term insurance</th>
<th>Pure endowment</th>
<th>Endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>7.879</td>
<td>70.960</td>
<td>78.839</td>
</tr>
<tr>
<td>Unit-linked call</td>
<td>0.342</td>
<td>2.652</td>
<td>2.994</td>
</tr>
<tr>
<td>Unit-linked guarantee</td>
<td>8.221</td>
<td>73.612</td>
<td>81.833</td>
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</tbody>
</table>

**Table 2.4**: Annual net premiums

<table>
<thead>
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<th>Product</th>
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<th>Pure endowment</th>
<th>Endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>0.908</td>
<td>8.179</td>
<td>9.087</td>
</tr>
<tr>
<td>Unit-linked call</td>
<td>0.039</td>
<td>0.306</td>
<td>0.345</td>
</tr>
<tr>
<td>Unit-linked guarantee</td>
<td>0.947</td>
<td>8.485</td>
<td>9.432</td>
</tr>
</tbody>
</table>

**Table 2.5**: Actuarial reserves

\[
\begin{array}{cccccccc}
 t & V^{T I, R}_{s} & V^{P E, R}_{s} & V^{E, R}_{s} & V^{T I C, R}_{s} & V^{P E C, R}_{s} & V^{E C, R}_{s} & V^{E G, R}_{s} \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0.341 & 8.433 & 8.774 & 0.022 & 0.471 & 0.493 & 9.267 \\
 2 & 0.631 & 17.140 & 17.771 & 0.039 & 0.962 & 1.001 & 18.772 \\
 3 & 0.864 & 26.140 & 27.004 & 0.053 & 1.469 & 1.522 & 28.526 \\
 4 & 1.031 & 35.458 & 36.489 & 0.061 & 1.986 & 2.047 & 38.536 \\
 5 & 1.122 & 45.123 & 46.245 & 0.064 & 2.503 & 2.567 & 48.812 \\
 6 & 1.126 & 55.167 & 56.293 & 0.060 & 2.997 & 3.057 & 59.350 \\
 7 & 1.031 & 65.629 & 66.660 & 0.051 & 3.425 & 3.476 & 70.136 \\
 8 & 0.823 & 76.552 & 77.375 & 0.035 & 3.692 & 3.727 & 81.102 \\
 9 & 0.485 & 87.989 & 88.474 & 0.016 & 3.529 & 3.545 & 92.019 \\
\end{array}
\]

**Table 2.6**: Risk premiums

\[
\begin{array}{cccccccc}
 t & \pi^{R, T I}_{s} & \pi^{R, P E}_{s} & \pi^{R, E}_{s} & \pi^{R, T I C}_{s} & \pi^{R, P E C}_{s} & \pi^{R, E C}_{s} & \pi^{R, E G}_{s} \\
 0 & 0.576 & -0.049 & 0.527 & 0.018 & -0.154 & -0.136 & 0.391 \\
 1 & 0.633 & -0.109 & 0.524 & 0.023 & -0.161 & -0.138 & 0.386 \\
 2 & 0.696 & -0.184 & 0.512 & 0.027 & -0.166 & -0.139 & 0.373 \\
 3 & 0.766 & -0.275 & 0.491 & 0.033 & -0.164 & -0.131 & 0.360 \\
 4 & 0.845 & -0.385 & 0.460 & 0.038 & -0.150 & -0.112 & 0.348 \\
 5 & 0.931 & -0.520 & 0.411 & 0.044 & -0.115 & -0.071 & 0.340 \\
 6 & 1.028 & -0.682 & 0.346 & 0.050 & -0.039 & 0.011 & 0.357 \\
\end{array}
\]
<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tau_{x,t}^{S,TL}$</th>
<th>$\tau_{x,t}^{S,PE}$</th>
<th>$\tau_{x,t}^{S,E}$</th>
<th>$\tau_{x,t}^{S,TIC}$</th>
<th>$\tau_{x,t}^{S,PEC}$</th>
<th>$\tau_{x,t}^{S,EC}$</th>
<th>$\tau_{x,t}^{S,EG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.332</td>
<td>8.228</td>
<td>8.560</td>
<td>0.021</td>
<td>0.459</td>
<td>0.481</td>
<td>9.041</td>
</tr>
<tr>
<td>1</td>
<td>0.275</td>
<td>8.288</td>
<td>8.563</td>
<td>0.017</td>
<td>0.467</td>
<td>0.484</td>
<td>9.047</td>
</tr>
<tr>
<td>2</td>
<td>0.212</td>
<td>8.363</td>
<td>8.575</td>
<td>0.012</td>
<td>0.471</td>
<td>0.483</td>
<td>9.058</td>
</tr>
<tr>
<td>3</td>
<td>0.142</td>
<td>8.454</td>
<td>8.596</td>
<td>0.007</td>
<td>0.469</td>
<td>0.476</td>
<td>9.072</td>
</tr>
<tr>
<td>4</td>
<td>0.064</td>
<td>8.564</td>
<td>8.628</td>
<td>0.001</td>
<td>0.456</td>
<td>0.457</td>
<td>9.085</td>
</tr>
<tr>
<td>5</td>
<td>−0.023</td>
<td>8.699</td>
<td>8.676</td>
<td>−0.005</td>
<td>0.421</td>
<td>0.416</td>
<td>9.092</td>
</tr>
<tr>
<td>6</td>
<td>−0.120</td>
<td>8.861</td>
<td>8.741</td>
<td>−0.011</td>
<td>0.345</td>
<td>0.334</td>
<td>9.075</td>
</tr>
<tr>
<td>7</td>
<td>−0.228</td>
<td>9.056</td>
<td>8.828</td>
<td>−0.016</td>
<td>0.177</td>
<td>0.161</td>
<td>8.989</td>
</tr>
<tr>
<td>8</td>
<td>−0.349</td>
<td>9.291</td>
<td>8.942</td>
<td>−0.019</td>
<td>−0.249</td>
<td>−0.268</td>
<td>8.674</td>
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</tbody>
</table>

Table 2.8: Sums at risk

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tau_{x,t}^{S,TL}$</th>
<th>$\tau_{x,t}^{S,PE}$</th>
<th>$\tau_{x,t}^{S,E}$</th>
<th>$\tau_{x,t}^{S,TIC}$</th>
<th>$\tau_{x,t}^{S,PEC}$</th>
<th>$\tau_{x,t}^{S,EC}$</th>
<th>$\tau_{x,t}^{S,EG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99.369</td>
<td>−17.140</td>
<td>82.229</td>
<td>3.541</td>
<td>−25.348</td>
<td>−21.807</td>
<td>60.422</td>
</tr>
<tr>
<td>3</td>
<td>98.969</td>
<td>−35.458</td>
<td>63.511</td>
<td>4.199</td>
<td>−21.128</td>
<td>−16.929</td>
<td>46.582</td>
</tr>
<tr>
<td>4</td>
<td>98.878</td>
<td>−45.123</td>
<td>53.755</td>
<td>4.475</td>
<td>−17.571</td>
<td>−13.096</td>
<td>40.659</td>
</tr>
<tr>
<td>5</td>
<td>98.874</td>
<td>−55.167</td>
<td>43.707</td>
<td>4.696</td>
<td>−12.237</td>
<td>−7.541</td>
<td>36.166</td>
</tr>
<tr>
<td>6</td>
<td>98.969</td>
<td>−65.629</td>
<td>33.340</td>
<td>4.838</td>
<td>−3.739</td>
<td>1.099</td>
<td>34.439</td>
</tr>
<tr>
<td>7</td>
<td>99.177</td>
<td>−76.552</td>
<td>22.625</td>
<td>4.857</td>
<td>11.259</td>
<td>16.116</td>
<td>38.741</td>
</tr>
<tr>
<td>8</td>
<td>99.515</td>
<td>−87.989</td>
<td>11.526</td>
<td>4.658</td>
<td>43.864</td>
<td>48.522</td>
<td>60.048</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>−100</td>
<td>0</td>
<td>3.990</td>
<td>−1037.43</td>
<td>−1033.44</td>
<td>−1033.44</td>
</tr>
</tbody>
</table>

Table 2.9: Number of shares and expected call-option payment

<table>
<thead>
<tr>
<th>$t$</th>
<th>$N(t)$</th>
<th>$E[(N(t)S(t) - G(t))_{+}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>93.458</td>
<td>5.979</td>
</tr>
<tr>
<td>2</td>
<td>87.344</td>
<td>8.447</td>
</tr>
<tr>
<td>3</td>
<td>81.630</td>
<td>10.336</td>
</tr>
<tr>
<td>4</td>
<td>76.290</td>
<td>11.924</td>
</tr>
<tr>
<td>5</td>
<td>71.299</td>
<td>13.318</td>
</tr>
<tr>
<td>6</td>
<td>66.634</td>
<td>14.576</td>
</tr>
<tr>
<td>7</td>
<td>62.275</td>
<td>15.729</td>
</tr>
<tr>
<td>8</td>
<td>58.201</td>
<td>16.800</td>
</tr>
<tr>
<td>9</td>
<td>54.393</td>
<td>17.802</td>
</tr>
<tr>
<td>10</td>
<td>50.835</td>
<td>18.748</td>
</tr>
</tbody>
</table>
3. **Traditional versus guaranteed unit-linked endowment insurance.**

The purpose of the present Section is two-fold. First, in Section 3.1 we show by example how the distribution of the random multi-period aggregate loss of a life portfolio and the associated VaR and CVaR measures can be exactly evaluated and well approximated by the quantities obtained from a gamma distribution assumption for the insurance risk. Then, Section 3.2 is devoted to a comparison of the traditional endowment insurance with the guaranteed unit-linked endowment insurance. Within a value based management context, it is shown that the guaranteed unit-linked contract performs better than the traditional one on a risk-adjusted return scale. Indeed, the expected gain per unit of standard deviation, VaR or CVaR of a portfolio of identical policies is greater for the guaranteed unit-linked product than for the traditional one.

3.1. **Gamma approximation of the insurance risk.**

To determine the VaR and CVaR measures of a life portfolio, it is necessary to evaluate the distribution of the aggregate loss in Chapter XI, formula (2.7). We follow the method applied in Hürlimann(2002c). The random accumulated return \( R_T \) on investment is modeled by a lognormal distribution with parameters \( \ln(1 + j) - \frac{1}{2} \sigma^2 \cdot T \) and \( \sigma \sqrt{T} \), with \( j \) and \( \sigma \) as in Section 2.1. The random accumulated aggregate claims \( S = \sum_{t=1}^{T} r^{T-t} \cdot S_t \) follows a multi-period individual life model, whose distribution can be evaluated using the two stage recursive formulas of De Pril(1989) (see Hürlimann(2002c) for details). As a simple but reliable approximation of \( S \), one first replaces \( S_1, ..., S_T \) by independent copies \( S^\perp_1, ..., S^\perp_T \), and thus replace \( S \) by \( S^\perp = \sum_{t=1}^{T} r^{T-t} \cdot S^\perp_t \). By Hürlimann(2002c), Section 4, this implies the stop-loss order relation \( S \leq S^\perp \), which means that \( S \) has been replaced by the slightly more dangerous \( S^\perp \) for which one has in particular \( E[S] = E[S^\perp] \) and \( Var[S] \leq Var[S^\perp] \). Then one approximates \( S^\perp \) by a gamma distributed random variable with mean \( E[S^\perp] \) and variance \( Var[S^\perp] \). As the following illustration shows, the obtained very tractable approximation yields enough accurate upper bounds for VaR and CVaR calculations. Consider a portfolio of \( N = 100 \) traditional endowment policies with the characteristics of Section 2.4. The parameters of the gamma approximation are determined by the formulas

\[
E[S^\perp] = N \cdot \sum_{k=1}^{T} \left( r^{T-k} k S^E_{x+k-1} \right) \cdot p_x q_x \cdot (1 - p_{x+k-1}), \tag{3.1}
\]

\[
Var[S^\perp] = N \cdot \sum_{k=1}^{T} \left( r^{T-k} k S^E_{x+k-1} \right)^2 \cdot p_x q_x \cdot (1 - p_{x+k-1}), \tag{3.2}
\]

where second order probabilities of death are used as in Section 2.4, and the sum at risk \( k S^E_x \) is found in Table 2.8. By definition of the insurance margin in Section X.2, one has \( M^{\text{ins}} = \frac{1}{\alpha} E[S^\perp] \) (recall that one assumes that 60% of the probabilities in the tariff suffice to cover the claims). Comparisons of exact and approximated values of VaR and CVaR for the confidence level \( \alpha = 0.99 \) are summarized in the Table 3.1.
Table 3.1: Comparison of VaR and CVaR measures

<table>
<thead>
<tr>
<th>model</th>
<th>VaR</th>
<th>CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>exact distribution</td>
<td>5409.59</td>
<td>5749.48</td>
</tr>
<tr>
<td>gamma approximation</td>
<td>5411.53</td>
<td>6032.15</td>
</tr>
</tbody>
</table>

### 3.2. Risk-adjusted performance measurement.

The use of capital requirements using ERC risk measures like VaR and CVaR is of great importance in risk-adjustment performance measurement and has implications for the value based management of insurance products. Consider the random return per unit of ERC to a fixed confidence level $\alpha$ of a life portfolio at time $T$, called *ERC gain ratio*, which is defined by

$$GR^\alpha_T := \frac{G_T}{ERC^\alpha_T[-G_T]}.$$

The expected value of the ERC gain ratio measures the *risk-adjusted return on capital*. This way of computing the return is commonly called RAROC (e.g. Matten(1996), p.59), and is here defined by

$$RAROC^\alpha[G_T] = E[GR^\alpha_T].$$

Related discussions of RAROC are found in the Chapters VIII, IX and in Hürlimann(2001f). The use of RAROC as a value based management tool is straightforward. Indeed, if a product manager has to decide upon the more profitable of two life portfolios with accumulated aggregate surplus $G^1_T$ and $G^2_T$, a decision in favor of the first portfolio is taken if and only if one has

$$RAROC^\alpha[G^1_T] \geq RAROC^\alpha[G^2_T]$$

at given confidence levels $\alpha$. This preference criterion tells us that a portfolio is preferred to another if its expected gain per unit of economic risk capital is greater. As an illustration we compare portfolios of $N$ identical policies with traditional endowment and guaranteed unit-linked endowment contracts under the assumptions of Section 2.4. We use a gamma approximation for the insurance risk as in Section 3.1 and note that $-S$ has to be assumed gamma distributed when $E[S] < 0$ as for our guaranteed unit-linked policy. Our results for the confidence level $\alpha = 0.99$ by varying portfolio size $N$ are summarized in the Tables 3.2 and 3.3. It is remarkable that the guaranteed unit-linked contract performs better than the traditional contract on the RAROC scale. One observes that the same property holds if the economic risk capital measure is replaced by the simpler standard deviation risk measure.

Table 3.2: Measurement of traditional and guaranteed unit-linked endowment life insurance

<table>
<thead>
<tr>
<th>N</th>
<th>$\mu_c$</th>
<th>$\sigma_c$</th>
<th>VaR</th>
<th>CVaR</th>
<th>$\mu_G$</th>
<th>$\sigma_G$</th>
<th>VaR</th>
<th>CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>5884</td>
<td>8057</td>
<td>5412</td>
<td>6032</td>
<td>6441</td>
<td>8445</td>
<td>5621</td>
<td>6420</td>
</tr>
<tr>
<td>200</td>
<td>11769</td>
<td>16112</td>
<td>10810</td>
<td>12095</td>
<td>12882</td>
<td>16806</td>
<td>10978</td>
<td>12512</td>
</tr>
<tr>
<td>500</td>
<td>29402</td>
<td>40277</td>
<td>27004</td>
<td>30283</td>
<td>32206</td>
<td>41889</td>
<td>26858</td>
<td>30524</td>
</tr>
<tr>
<td>1000</td>
<td>58845</td>
<td>80553</td>
<td>53994</td>
<td>60597</td>
<td>64412</td>
<td>83694</td>
<td>53231</td>
<td>60381</td>
</tr>
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</table>
Table 3.3: Comparison of risk-adjusted performance measures

<table>
<thead>
<tr>
<th>N</th>
<th>$\mu_G$</th>
<th>$\sigma_G$</th>
<th>VaR</th>
<th>CVaR</th>
<th>$\mu_G$</th>
<th>$\sigma_G$</th>
<th>VaR</th>
<th>CVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.731</td>
<td>1.087</td>
<td>0.976</td>
<td>0.763</td>
<td>1.003</td>
<td>1.146</td>
<td>1.003</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>0.731</td>
<td>1.089</td>
<td>0.973</td>
<td>0.767</td>
<td>1.030</td>
<td>1.174</td>
<td>1.030</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>0.731</td>
<td>1.090</td>
<td>0.972</td>
<td>0.764</td>
<td>1.055</td>
<td>1.199</td>
<td>1.055</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.731</td>
<td>1.090</td>
<td>0.971</td>
<td>0.770</td>
<td>1.067</td>
<td>1.210</td>
<td>1.067</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX A

Gamma approximation of the aggregate claims distribution

Suppose the aggregate claims random variable of an insurance risk portfolio can be represented by a compound Poisson random variable

\[ S = \sum_{i=1}^{N} Y_i, \quad (A.1) \]

where \( N \) is Poisson(\( \lambda \)) distributed, the \( Y_i \)'s are independent and identically distributed non-negative random variables, which are stochastically independent from \( N \). Denote the identical random variables by \( Y = Y_i, \quad i = 1, \ldots, N \). From a practical viewpoint, it has been stated for a long time that a gamma approximation of the claim size is appropriate for modeling the insurance risk process in life insurance (e.g. OECD(1971), Strickler(1982), Drude(1988), p.183, Hürlimann(1988b)). Theoretically, this claim size model arises as unique solution of a characterization problem for scale compound parametric families of distributions with the mean as scale parameter (Proposition 3.2 in Hürlimann(1998c)). Let us follow this approach. A gamma claim size has density

\[ f_Y(x) = g(x; \alpha, \beta) = \frac{(\beta x)^{\alpha} e^{-\beta x}}{\Gamma(\alpha)} \cdot \frac{e^{-\beta x}}{x}, \quad x > 0, \quad (A.2) \]

with the parameters

\[ \alpha = \frac{m_2 - m_1^2}{m_1}, \quad \beta = \frac{\alpha}{m_1}, \quad m_i = E[Y], \quad i = 1, 2. \quad (A.3) \]

The distribution function of the claim size is given by the incomplete gamma function

\[ F_Y(x) = G(\beta x; \alpha) = \frac{1}{\Gamma(\alpha)} \int_0^{\beta x} t^{\alpha-1} e^{-t} dt. \quad (A.4) \]

Using Laplace transforms one sees that the \( j \)-th convolution of the claim size density equals

\[ f_Y^{*j}(x) = g(x; j\alpha, \beta), \quad j = 1, 2, \ldots. \quad (A.5) \]

It follows that the aggregate claims distribution function has the representation

\[ F_S(x; \lambda, \alpha, \beta) = e^{-\lambda} + \sum_{j=1}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} \cdot G(\beta x; j\alpha). \quad (A.6) \]

The following result justifies the use of the gamma approximation to the aggregate claims distribution for sufficiently large portfolios.
**Theorem A.1.** Under the above assumptions, the compound Poisson gamma distribution of the aggregate claims converges to a gamma distribution as the expected number of claims tend to infinity. More precisely, with the parameter function of one variable

\[ a(\lambda) = \left( k^2 - \lambda^{-1} \right)^{-1} \]

such that \( \alpha = \alpha(\lambda) = \lambda^{-1} \cdot a(\lambda) \), \( \beta = \beta(\lambda) = \mu^{-1} \cdot a(\lambda) \), the limiting distribution is given by

\[
\lim_{\lambda \to \infty} F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) = G(bx; a), \quad a = \frac{1}{k^2}, \quad b = \frac{1}{k^2 \mu}. \tag{A.7}
\]

**Proof.** A simple calculation shows the validity of the limit

\[
\lim_{\lambda \to \infty} \lambda f_x(x; \alpha(\lambda), \beta(\lambda)) = \lim_{\lambda \to \infty} \left\{ \frac{a(\lambda) \beta(\lambda)^{\alpha(\lambda)} e^{-\beta(\lambda)x}}{\Gamma(\alpha(\lambda) + 1)} \cdot x^{\alpha(\lambda)} \cdot e^{-\beta(\lambda)x} \right\} = \frac{a}{x} e^{-bx}. \tag{A.8}
\]

Using Dufresne et al. (1991), Section 2, this asymptotic result identifies \( \lim_{\lambda \to \infty} F_S(x) \) with a gamma process, whose distribution function equals \( G(bx; a) \).

\[ \diamondsuit \]

**APPENDIX B**

**Value-at-risk upper bound**

As in Appendix A, Theorem A.1, suppose the aggregate claims random variable \( S \) has the compound Poisson gamma distribution

\[
F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) = e^{-\lambda x} + \sum_{j=1}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \cdot G(\beta(\lambda)x; j\alpha(\lambda)), \tag{B.1}
\]

which should be compared with the limiting gamma distribution \( G(bx; a) \). Our analysis applies standard results from ordering of risks theory (e.g. Kaas et al. (1994)).

**Definitions B.1.** Let \( X, Y \) be random variables with distribution functions \( F_X(x), F_Y(x) \) and densities \( f_X(x), f_Y(x) \). Consider the following stochastic orders:

- (SD) \( X \) precedes \( Y \) in the *stochastic dominance of first order*, written \( X \leq_{SD} Y \), if \( F_X(x) \geq F_Y(x) \) for all \( x \) in the common support of \( X \) and \( Y \).
- (LR) \( X \) precedes \( Y \) in the *likelihood ratio order*, written \( X \leq_{LR} Y \), if the ratio of likelihoods \( f_X(x)/f_Y(x) \) is a decreasing function on the common support of \( X \) and \( Y \).

It is well-known that \( X \leq_{LR} Y \Rightarrow X \leq_{SD} Y \) (Kaas et al. (1994), Section V.1).

First, we compare the gamma distributed random variables

\[
X_j \sim \Gamma(j\alpha(\lambda), \beta(\lambda)), \quad j = 1, 2, \ldots \tag{B.2}
\]

\[
X \sim \Gamma(a, b) \tag{B.3}
\]
where we suppose that $\alpha = \alpha(\lambda), \beta = \beta(\lambda) > 0$ (this holds for sufficiently large $\lambda$).

**Lemma B.1.** The random variables $X_j$ and $X$ satisfy the following relationships:

- **Case 1:** if $j \leq \lambda - k^2$ then $X_j \leq_{LR} X$, hence $X_j \leq_{SD} X$.
- **Case 2:** if $j > \lambda - k^2$ there exists a constant $c(j)$ such that the tails satisfy the relation $F_{X_j}(x) \geq F_X(x)$ for all $x \geq c(j)$.

**Proof.** Let $f_j(x) = g(x; j\alpha, \beta)$ and $f(x) = g(x; a, b)$ be the gamma densities of $X_j$ and $X$. The likelihood ratio $q(x) = f_j(x) / f(x)$ has the derivative

$$q'(x) = \frac{\Gamma(a)\beta^{ja}}{\Gamma(j\alpha)b^a} x^{j\alpha - 1} e^{-(\beta - b)x} \cdot [(j\alpha - a) - (\beta - b)x].$$

(B.4)

It is not difficult to see that

$$\beta - b = \frac{1}{\mu} \left[ a(\lambda) - \frac{1}{k^2} \right] > 0, \quad j\alpha - a = \frac{j}{\lambda} a(\lambda) - \frac{1}{k^2} \leq 0 \iff j \leq \frac{\lambda}{k^2}.$$  

(B.5)

Therefore, in Case 1 the derivative is negative, hence $q(x)$ is decreasing, which shows that $X_j \leq_{LR} X$. In Case 2 the function $q(x)$ has a maximum $q(x_0) > 1$ at $x_0 = \frac{j\alpha - a}{\beta - b}$, and the difference $\Delta(x) = f_j(x) - f(x) = f(x) \cdot [q(x) - 1]$ has two sign changes in the order $-\rightarrow +\rightarrow -$. It follows that $F_j(x) - F(x)$ has one sign change from $-$ to $+$ (e.g. Kaas et al. (1994), proof of Theorem III.1.4). ◊

Secondly, we compare the tails of the compound Poisson gamma distribution and its limiting gamma approximation.

**Theorem B.1.** Under the assumptions of Theorem A.1 one has the asymptotic inequality

$$\lim_{x \to \infty} F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) \geq e^{-\lambda} + (1 - e^{-\lambda}) \cdot \lim_{x \to \infty} G(bx; a)$$

(B.6)

for all sufficiently high $\lambda$ with $\alpha(\lambda), \beta(\lambda) > 0$.

**Proof.** Using the infinite series gamma representation of $F_S$ and the Lemma B.1, there exists for all integers $N \geq 1$ a constant $c(N)$ such that

$$F_S(x; \lambda, \alpha(\lambda), \beta(\lambda)) \geq e^{-\lambda} + \left( \sum_{j=1}^{N} e^{-\lambda} \frac{\lambda^j}{j!} \right) \cdot G(bx; a)$$

(B.7)

for all $x \geq c(N)$. Taking limits as $x \to \infty$ and $N \to \infty$ shows the assertion. ◊

The preceding result implies that for sufficiently high confidence levels $\alpha$ and sufficiently large $\lambda$ the value-at-risk of the aggregate claims random variable $S$ is bounded above by
VaR_a[S] ≤ VaR_a[X], where X is the gamma approximation to S. This inequality yields also an upper bound for ERC as defined in the text. For a numerical example, set $\mu = 0.7$, $k = 0.2$, $\lambda = 100$ and $\alpha = 0.99$. One obtains the upper bound $ERC_a[S] = VaR_a[S] - \mu = 0.361 \leq ERC_a[X] = VaR_a[X] - \mu = 0.366$. Since the parameters are subject to a considerable estimation uncertainty (especially for the coefficient of variation), the proposed simple gamma approximation can be recommended for practical purposes.

APPENDIX C

Stochastic ordered bounds by known moments to order four

To construct the stochastic and stop-loss ordered maximal distribution functions $F_{tr,\max}^{(n)}(x)$ and $F_{sl,\max}^{(n)}(x)$ for the moments spaces $D_n := D_n([A,B],\mu_1,\ldots,\mu_n)$, it is necessary to solve the optimization problems $\max_{x \in D_n} \{ \mathbb{E}[I_{[x,\infty)}(X)] \}$ and $\max_{x \in D_n} \{ \mathbb{E}[(X - x)_+] \}$, where $I_{[x,\infty)}(z)$ is the Heaviside indicator function, defined to be 0 if $z < x$ and 1 otherwise. Both belong to the class of extremal problems $\max_{x \in D_n} \{ \mathbb{E}[f(X)] \}$, where $f(x)$ is a piecewise linear function, and which have been extensively studied in Hürlimann (1997b/c/98e). A general approach to solve these problems is the well-known polynomial majorant method, which consists to bound $f(x)$ by some polynomial $q(x)$ of degree less or equal to $n$, and to construct a finite atomic random variable $Z \in D_n$ such that all atoms of $f(Z)$ are simultaneously atoms of $q(Z)$. Indeed, suppose $q(x)$ and $Z$ have been found such that $\Pr(q(Z) = f(Z)) = 1$ and $q(x) \geq f(x)$ for all $x \in [A,B]$. Then the expected value $\mathbb{E}[q(Z)] = \mathbb{E}[f(Z)]$ depends only on the first moments $\mu_1,\ldots,\mu_n$, and thus necessarily $Z$ maximizes $\mathbb{E}[f(X)]$ over all $X \in D_n$. A brief outline of the Appendix C follows.

In Section 1 we derive in a first step finite exhaustive lists of all polynomials of a given degree, which can be used to construct polynomial majorants for $f(x) = I_{[x,\infty)}(x)$ and $f(x) = (x - d)_+$. A second step in the construction of best bounds for expected values by given range and known moments of higher order consists in a detailed analysis of the algebraic moment problem for finite atomic random variables. The most useful results are based on the explicit analytical structure of bi- and triatomic random variables by given range and known moments up to order four as presented in Section 2. Then, in Section 3 we derive the stochastic ordered bounds and in Section 4 the corresponding stop-loss ordered bounds.

1. Polynomial majorants for the Heaviside indicator and the stop-loss functions.

The Heaviside indicator function $f(x) = I_{[x,\infty)}(x)$ and the stop-loss function $f(x) = (x - d)_+$, $d$ the deductible, belong to the class of piecewise linear functions $f(x)$ on an interval $I = [a,b]$, $-\infty \leq a < b \leq \infty$. For these simple but most important prototypes, one can decompose $I$ into two disjoint adjacent pieces such that $I = I_1 \cup I_2$, and the function of interest is a linear function $f(x) = \ell_i(x) = \alpha_i + \beta_i x$ on each piece $I_i$, $i = 1,2$. If $q(x)$ is a polynomial of degree $n \geq 2$, then $q(x) - f(x)$ is a piecewise polynomial function of degree
$n$, which is denoted by $Q(x)$ and which coincides on $I_i$ with the polynomial $Q_i(x) = q(x) - \ell_i(x)$ of degree $n$. For the construction of polynomial majorants $q(x) \geq f(x)$ on $I$, one can restrict the attention to finite atomic random variables $X$ with support $\{x_0 = a, x_1, \ldots, x_n, x_{n+1} = b\} \subset \{a, b\}$ such that $Pr(q(X) = f(X)) = 1$ (e.g. Karlin and Studden(1966), Theorem XII.2.1). By convention, if $a = -\infty$ then $x_0 = a$ is removed from the support and if $b = \infty$ then $x_{n+1} = b$ is removed. In general, the fact that $x_0 = a$ or/and $x_{n+1} = b$ does no belong to the support of $X$ is technically achieved by setting the corresponding probabilities equal to zero. If an atom of $X$, say $x_k$, is an interior point of some $I_i$, then it must be a double zero of $Q_i(x)$. Indeed $q(x) \geq \ell_i(x)$ for $x \in I_i$ is only fulfilled if the line $\ell_i(x)$ is tangent to $q(x)$ at $x_k$, that is $q'(x_k) = \ell_i'(x_k)$. From this simple observation, one derives finite exhaustive lists of all polynomials of a given degree, which are used to construct polynomial majorants (Tables 1.1 and 1.2).

Consider first the indicator function $f(x) = I_{[a,b]}(x)$. We decompose the interval $I = [a,b]$ into the pieces $I_1 = [a,t], I_2 = [t,b]$, such that $f(x) = \ell_1(x) = 0$ on $I_1$ and $f(x) = \ell_2(x) = 1$ on $I_2$. For a fixed $m \in \{1, \ldots, r\}$ the atom $x_m = t$ belongs always to the support of a maximizing finite atomic random variable $X$. We show that a polynomial majorant of fixed degree is always among the finite many possibilities listed in Table 1.1.

**Proposition 1.1.** Let $\{x_0 = a, x_1, \ldots, x_n = t, \ldots, x_r, x_{r+1} = b\}$, $x_r < x_s$ for $r < s$, $m \in \{1, \ldots, r\}$, be the support of a random variable $X$ on $I$, and let $q(x)$ be a polynomial majorant such that $Pr(q(X) = f(X)) = 1$ and $q(x) \geq f(x)$ on $I$. Then $q(x)$ is uniquely determined by the conditions in Table 1.1.

**Table 1.1**: polynomial majorants for the Heaviside indicator function

<table>
<thead>
<tr>
<th>case</th>
<th>support with $x_m = t$</th>
<th>$Q_i(x_j) = 0, j = 1,2$</th>
<th>$Q_j(x_i) = 0, j = 1,2$</th>
<th>$\deg q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>${a, x_1, \ldots, x_n, b}$</td>
<td>$i = 0, \ldots, r+1$</td>
<td>$i \neq 0, m, r+1$</td>
<td>$2r$</td>
</tr>
<tr>
<td>(2)</td>
<td>${a, x_1, \ldots, x_n}$</td>
<td>$i = 0, \ldots, r$</td>
<td>$i \neq 0, m$</td>
<td>$2r - 1$</td>
</tr>
<tr>
<td>(3)</td>
<td>${x_1, \ldots, x_n, b}$</td>
<td>$i = 1, \ldots, r+1$</td>
<td>$i \neq m, r+1$</td>
<td>$2r - 1$</td>
</tr>
<tr>
<td>(4)</td>
<td>${x_1, \ldots, x_n}$</td>
<td>$i = 1, \ldots, r$</td>
<td>$i \neq m$</td>
<td>$2r - 2$</td>
</tr>
</tbody>
</table>

**Proof.** Restrict the attention to case (1) (the other cases are shown similarly). One must show the existence of a unique polynomial $q(x)$ of degree $n = 2r$ as in Figure 1.1.

**Figure 1.1**: polynomial majorant $q(x) \geq I_{[a,b]}(x)$, $x \in [a,b]$
Consider the unique polynomial \( q(x) \) of degree \( n = 2r \) such that

\[
q(x) = \begin{cases} 
0, & i = 0, \ldots, m-1 \\
1, & i = m, \ldots, r+1 \end{cases}, \quad q'(x) = 0, \quad i \neq 0, m, r+1.
\]

By definition of \( Q_j(x), j = 1,2 \), the conditions of Table 1.1 under case (1) are fulfilled. By the theorem of Rolle, the derivative \( q'(x) \) vanishes at least once on each of the \( r \) subintervals \((x_{i-1}, x_i)\), \( 0 \leq i \leq r, i \neq m-1 \). In fact one has exactly \((r-1) + r = n-1\) zeros of \( q'(x) \) on \( I \). Furthermore, one has \( q'(x) \neq 0 \) on \((x_{m-1}, x_m)\). More precisely, one has \( q'(x) > 0 \) on \((x_{m-1}, x_m)\) because \( q(x_{m-1}) = 0 < q(x_m) = 1 \). It follows that \( q(x) \) is local minimal at all \( x_i, i \neq 0, m, r+1, \) and local maximal between each consecutive minima, as well as in the intervals \((a, x_i)\) and \((x_i, b)\). These properties imply the inequality \( q(x) \geq I_{[a,b]}(x), \ x \in [a,b] \).

Consider now the stop-loss function \( f(x) = (x-d) \),  \( d \in (a,b) \) the deductible, \( I_1 = [a,d], \ I_2 = [d,b] \). Then the stop-loss function \( f(x) \) may be viewed as the piecewise linear function defined by \( f(x) = \ell_1(x) = 0 \) on \( I_1 \), \( f(x) = \ell_2(x) = x-d \) on \( I_2 \). By convention \( m \in \{1, \ldots, r\} \) is fixed such that \( x_m < d < x_{m+1} \). A polynomial majorant of fixed degree for \( f(x) \) belongs always to one of the finitely many types listed in Table 1.2 below. The notations \( Q(x; x_1, d) \) and \( Q_j(x; x_1, d) \) mean that these functions depend upon the parameter vector \( \xi = (x_0, \ldots, x_{r+1}) \) and the deductible \( d \).

**Proposition 1.2.** Let \( \{x_0 = a, x_1, \ldots, x_r = b\} \), \( x_r < x_s \) for \( r < s \), \( x_m < d < x_{m+1} \), be the ordered support of a random variable \( X \) defined on \( I \), and let \( q(x) \) be a polynomial majorant such that \( \Pr(q(X) = f(X)) = 1 \) and \( q(x) \geq f(x) \) on \( I \). Then \( q(x) \) is a polynomial uniquely determined by the conditions in Table 1.2.

**Table 1.2:** Polynomial majorants for the stop-loss function

<table>
<thead>
<tr>
<th>Case</th>
<th>Support</th>
<th>( Q_i(x_1) = 0 ), ( j = 1,2 )</th>
<th>( Q_j(x_1) = 0 ), ( j = 1,2 )</th>
<th>Degree</th>
<th>Condition on deductible ( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a)</td>
<td>( {a, x_1, \ldots, x_r} )</td>
<td>( i = 0, \ldots, r+1 )</td>
<td>( i = 1, \ldots, r-1 )</td>
<td>2r</td>
<td>( Q_2(x; \xi, d) = 0 )</td>
</tr>
<tr>
<td>(1b)</td>
<td>( {a, x_1, \ldots, x_r} )</td>
<td>( i = 0, \ldots, r+1 )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r+1</td>
<td>( Q_1(y; \xi, d) = 0, \ y \in (\xi, a] )</td>
</tr>
<tr>
<td>(1c)</td>
<td>( {a, x_1, \ldots, x_r} )</td>
<td>( i = 0, \ldots, r+1 )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r+1</td>
<td>( Q_2(z; \xi, d) = 0, \ z \in [b, \infty) )</td>
</tr>
<tr>
<td>(2a)</td>
<td>( {x_1, \ldots, x_r} )</td>
<td>( i = 1, \ldots, r+1 )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r</td>
<td>( Q_2(z; \xi, d) = 0, \ z \in [b, \infty) )</td>
</tr>
<tr>
<td>(2b)</td>
<td>( {x_1, \ldots, x_r} )</td>
<td>( i = 1, \ldots, r+1 )</td>
<td>( i = 2, \ldots, r )</td>
<td>2r-1</td>
<td>( Q_1(x_1; \xi, d) = 0 )</td>
</tr>
<tr>
<td>(3a)</td>
<td>( {a, x_1, \ldots, x_r} )</td>
<td>( i = 0, \ldots, r )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r-1</td>
<td>( Q_1(y; \xi, d) = 0, \ y \in (\xi, a] )</td>
</tr>
<tr>
<td>(3b)</td>
<td>( {a, x_1, \ldots, x_r} )</td>
<td>( i = 0, \ldots, r )</td>
<td>( i = 1, \ldots, r-1 )</td>
<td>2r-2</td>
<td>( Q_2(x_1; \xi, d) = 0 )</td>
</tr>
<tr>
<td>(4a)</td>
<td>( {x_1, \ldots, x_r} )</td>
<td>( i = 1, \ldots, r )</td>
<td>( i = 1, \ldots, r-1 )</td>
<td>2r-1</td>
<td>( Q_2(y; \xi, d) = 0, \ y \in (\xi, a] )</td>
</tr>
<tr>
<td>(4b)</td>
<td>( {x_1, \ldots, x_r} )</td>
<td>( i = 1, \ldots, r )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r-1</td>
<td>( Q_2(z; \xi, d) = 0, \ z \in [a, \infty) )</td>
</tr>
<tr>
<td>(4c)</td>
<td>( {x_1, \ldots, x_r} )</td>
<td>( i = 1, \ldots, r )</td>
<td>( i = 1, \ldots, r )</td>
<td>2r-1</td>
<td>( Q_2(z; \xi, d) = 0, \ z \in [a, \infty) )</td>
</tr>
</tbody>
</table>
There are essentially two typical cases for which a proof is required, say (1a) and (1b). The other cases are shown by the same method and omitted for this reason. In case (1a) one shows the existence of a unique polynomial \( q(x) \) of degree \( n = 2r \) as in Figure 1.2.

Consider the unique polynomial \( q(x) \) of degree \( n = 2r \) such that

\[
q(x) = \begin{cases} 
0, & i = 0, \ldots, m \\
x_i - d, & i = m + 1, \ldots, r + 1
\end{cases} 
q'(x) = \begin{cases} 
0, & i = 1, \ldots, m \\
1, & i = m + 1, \ldots, r - 1
\end{cases}
\]

Figure 1.2: polynomial majorant \( q(x) \geq (x - d)_+ \), \( x \in [a, b] \), case (1a)

By definition of \( Q_j(x) \), \( j = 1, 2 \), the conditions of Table 1.2 under case (1a) are fulfilled. In order that \( q(x) \geq (x - d)_+ \), the line \( \ell_2(x) = x - d \) must be tangent of \( q(x) \) at the remaining atom \( x = x_i \), that is \( q'(x_i) = 1 \) or \( Q_2(x_i; \xi, d) = 0 \). This condition is an implicit equation for the deductible \( d \) and restricts its range of variation. The theorem of Rolle implies the following facts:

(i) \( Q_1(x) = q'(x) \) vanishes at least once on each of the \( m \) subintervals \( (x_i, x_{i+1}) \), \( i = 1, \ldots, m \).

(ii) \( Q_2(x) = q'(x) - 1 \) vanishes at least once on each of the \( r - m \) subintervals \( (x_i, x_{i+1}) \), \( i = m + 1, \ldots, r \).

(iii) \( Q_1(x) \neq 0 \) on \( (x_m, d) \) and \( Q_2(x) \neq 0 \) on \( [d, x_{m+1}] \). More precisely one has \( Q_1(x) > 0 \) on \( (x_m, d) \) since \( Q_1(x_m) = 0 < Q_1(x_{m+1}) = x_{m+1} - d \), and \( Q_2(x) < 0 \) on \( [d, x_{m+1}] \) since \( Q_2(x_m) = d - x_m > 0 = Q_2(x_{m+1}) \).

In particular there are exactly \( n - 1 \) zeros of \( q'(x) \) on \( I \). It follows that \( Q_1(x) \) is local minimal at all \( x_i, \ i = 1, \ldots, m \), and local maximal between each consecutive minima, as well as in the interval \( (a, x_i) \). Similarly \( Q_2(x) \) is local minimal at all \( x_i, \ i = m + 1, \ldots, r \), and local maximal between each consecutive minima, as well as in the interval \( (x_r, b) \). These properties imply that \( Q_1(x) \geq 0 \) on \( I_1 \) and \( Q_2(x) \geq 0 \) on \( I_2 \), which together means that \( q(x) \geq (x - d)_+ \) on \( I_1 \cup I_2 = I \). In case (1b) one shows the existence of a unique polynomial \( q(x) \) of degree \( n = 2r + 1 \) as in Figure 1.3, where \( y \) is a further zero of \( q(x) \) in \( (-\infty, a] \).
Figure 1.3: polynomial majorant $q(x) \geq (x-d)_+$, $x \in [a,b]$, case (1b)

Consider the unique polynomial $q(x)$ of degree $n = 2r + 1$ such that

$$q(x_i) = \begin{cases} 0, & i = 0, \ldots, m \\ x_i - d, & i = m+1, \ldots, r+1 \end{cases}, \quad q'(x_i) = \begin{cases} 0, & i = 1, \ldots, m \\ 1, & i = m+1, \ldots, r \end{cases}.$$

By definition of $Q_j(x)$, $j = 1,2$, the conditions of Table 1.2 under case (1b) are fulfilled. In order that $q(x)$ is a polynomial of odd degree, there must exist a further zero $y$ of $q(x)$ in $(-\infty,a]$, which yields the implicit equation $Q_j(y; \xi; d) = 0$ for the deductible $d$. The rest of the proof follows similarly to case (1a). ♦

2. Structure of finite atomic random variables by known moments to order four.

The algebraic-analytical structure of the required sets of finite atomic random variables by given range and known moments to order four is implicit in Jansen et al. (1986), Section 2. However, by considering without loss of generality only standardized random variables, much calculation can been simplified and some results find improvement. Our derivation uses the solution of the algebraic moment problem by Mammana (1954), which itself is a direct application of the mathematical theory of orthogonal polynomials.

Consider a real random variable $X$ with an infinite number of non-zero finite moments $\mu_k = E[X^k]$, $k = 0,1,2,\ldots$. By convention one sets $p_0(x) = \mu_0 = 1$.

**Definition 2.1.** The orthogonal polynomial of degree $n \geq 1$ with respect to the moment structure $\{\mu_k\}_{k=0,\ldots,2n-1}$, also called orthogonal polynomial with respect to $X$, is the unique monic polynomial $p_n(x)$ of degree $n$, which satisfies the $n$ linear expected value equations

$$E[p_n(X) \cdot X^i] = 0, \quad i = 0,1,\ldots,n-1. \quad (2.1)$$

Note that the terminology "orthogonal" refers to the scalar product induced by the expectation operator $\langle X, Y \rangle = E[XY]$, where $X, Y$ are random variables for which this quantity exists. The orthogonal polynomials are also called classical Chebyshev polynomials.

Given the first $2n-1$ moments of some real random variable $X$, the algebraic moment problem of order $n$ (AMP(n)) asks for the existence and construction of a finite atomic
random variable with ordered support \{x_1, \ldots, x_n\} such that \( x_1 < x_2 < \ldots < x_n \), and probabilities \( \{p_1, \ldots, p_n\} \) such that the system of non-linear equations
\[
\sum_{i=1}^{n} p_i x_i^k = \mu_k, \quad k = 0, \ldots, 2n - 1, \tag{2.2}
\]
is solvable. For computational purposes it suffices to know that if a solution exists, then the atoms of the random variable solving AMP(n) must be identical with the distinct real zeros of the orthogonal polynomial of degree \( n \), as shown by the following precise recipe.

**Lemma 2.1.** (Mammana(1954)) Given are positive numbers \( p_1, \ldots, p_n \) and real distinct numbers \( x_1 < x_2 < \ldots < x_n \) such that the system AMP(n) is solvable. Then the \( x_i \)'s are the distinct real zeros of the orthogonal polynomial of degree \( n \), that is \( p_n(x_i) = 0, \quad i = 1, \ldots, n \), and
\[
p_i = \prod_{j \neq i} (x_i - x_j)^{-1} \cdot E \left[ \prod_{j \neq i} (Z - x_j) \right], \quad i = 1, \ldots, n, \tag{2.3}
\]
where \( Z \) denotes the discrete random variable with support \( \{x_1, \ldots, x_n\} \) and probabilities \( \{p_1, \ldots, p_n\} \) defined by AMP(n).

As a next preliminary step, it is important to state the conditions under which there exist random variables on a finite interval with given moments to order four (e.g. Jansen et al.(1986)). A recent general proof for the existence of moment spaces, or equivalently for the existence of random variables with known moments up to a given order, is in De Vylder(1996), part II, Chapter 3.3. From now on the attention is restricted to the set \( D(a,b) \) of all standard random variables with support \( [a,b] \), \(-\infty \leq a < b \leq \infty\), mean \( \mu = 0 \) and standard deviation \( \sigma = 1 \).

**Lemma 2.2.** (Moment inequalities) There exist non-degenerate standard random variables in \( D(a,b) \) if and only if the following two conditions hold:
\[
a < 0 < b \quad \text{(inequalities on the mean } \mu = 0 \text{)} \tag{2.4}
\]
\[
1 + ab \leq 0 \quad \text{(inequality on the variance } \sigma^2 = 1 \text{)} \tag{2.5}
\]

There exist non-degenerate standard random variables in \( D(-\infty, \infty) \) with given moments to order four if and only if the following inequality holds:
\[
\Delta = \gamma_2 - \gamma^2 + 2 \geq 0, \tag{2.6}
\]
where for \( X \in D(-\infty, \infty) \) the parameter \( \gamma = E[X^3] \) denotes the skewness and \( \gamma_2 = E[X^4] - 3 \) denotes the kurtosis.

**Proof.** The first two inequalities follow by taking expectations in the following random inequalities, which are valid with probability one for all \( X \in D(a,b) \):
\[
a \leq X \leq b, \tag{2.4'}
\]
\[(b - X)(X - a) \geq 0, \quad (2.5')\]

where for a non-degenerate random variable, the inequalities in \((2.4')\) must be strict. The inequality \((2.6)\) follows from the inequality

\[ (X - c)^2 \cdot (X - \bar{c})^2 \geq 0, \quad (2.6') \]

where \(c = \frac{1}{4}(\gamma - \sqrt{4 + \gamma^2}), \quad \bar{c} = -c^{-1} = \frac{1}{4}(\gamma + \sqrt{4 + \gamma^2}).\)

\[\Diamond\]

**Remark 2.1.** The inequality \((2.6)\) between skewness and kurtosis has been known for a long time (e.g. Pearson(1916), Wilkins(1944) and Guiard(1980)).

**Theorem 2.1.** \((Characterization of standard biatomic random variables on \([a, b]\))\) Suppose that \(a < 0 < b\) and \(1 + ab \leq 0\). The support \(\{x_1, x_2\}, \quad x_1 < x_2\), and probabilities \(\{p_1, p_2\}\) of a biatomic standard random variables \(X \in D(a, b)\) are uniquely determined by

\[ x_1 = x \in [a, \bar{b}], \quad x_2 = \bar{x} \in [\bar{a}, b], \quad (2.7) \]
\[ p_1 = \frac{1}{1 + x^2}, \quad p_2 = \frac{x^2}{1 + x^2}, \quad (2.8) \]

where the upper bar denotes a strictly increasing involution function such that \(\bar{x} = x\), which maps \(x \neq 0\) to \(\bar{x} = -1/x\).

**Proof.** By Lemma 2.2 the conditions \((2.4)\) and \((2.5)\) are required. By Lemma 2.1 (solution of AMP(2)) the atoms \(x_1, x_2\) are the distinct real zeros of the standard quadratic orthogonal polynomial \(p_2(x) = x^2 - \gamma x - 1\), where \(\gamma\) is a variable skewness parameter. The Vietà formulas imply the relations \(x_1 + x_2 = \gamma, \quad x_1 x_2 = -1\). Setting \(x_1 = x\) one must have \(x_2 = \bar{x}\) and \(x < 0 < \bar{x}\). If \(x \in (\bar{b}, 0)\) then \(\bar{x} \in (b, \infty)\) and the support \(\{x, \bar{x}\}\) is not feasible.

Therefore one must have \(x \in [a, \bar{b}]\), which determines uniquely the support \(\{x, \bar{x}\}\). The formulas \((2.8)\) for the probabilities follows from \((2.3)\), that is from

\[ p_1 = E\left[\frac{X - \bar{x}}{x - \bar{x}}\right], \quad p_2 = E\left[\frac{X - x}{\bar{x} - x}\right]. \quad \Diamond\]

**Proposition 2.2.** \((Characterization of standard triatomic random variables on \((-\infty, \infty)\) with skewness \(\gamma\) and kurtosis \(\gamma_2)\)\) Suppose that \(\Delta = \gamma_2 - \gamma^2 + 2 \geq 0\). The support \(\{x_1, x_2, x_3\}, \quad x_1 < x_2 < x_3\), and probabilities \(\{p_1, p_2, p_3\}\) of a triatomic random variable \(X \in D(-\infty, \infty)\) with skewness \(\gamma\) and kurtosis \(\gamma_2\), are uniquely determined as follows:

\[ x_1 = x \in (-\infty, c], \quad x_2 = \phi(x, \psi(x)) \in [c, \bar{c}], \quad x_3 = \psi(x) \in [\bar{c}, \infty), \quad (2.9) \]
\[ p_i = p(x_i), \quad i = 1, 2, 3, \quad (2.10) \]

where \(c = \frac{1}{4}(\gamma - \sqrt{4 + \gamma^2}), \quad \bar{c} = -c^{-1} = \frac{1}{4}(\gamma + \sqrt{4 + \gamma^2})\), and the functions \(\phi(u, v), \psi(u)\) and \(p(u)\) are defined by
\begin{equation}
\varphi(u,v) = \frac{v - u - v}{1 + uv},
\end{equation}

(2.11)

\begin{equation}
\psi(u) = \frac{1}{2} \left( \frac{A(u) - \sqrt{A(u)^2 + 4q(u)B(u)}}{q(u)} \right),
\end{equation}

(2.12)

\[ A(u) = \varphi(u) + \Delta u, \quad B(u) = \Delta + q(u), \quad q(u) = 1 + \gamma u - u^2, \]

(2.13)

\begin{equation}
p(u) = \frac{\Delta}{q(u)^2 + \Delta(1 + u^2)}.
\end{equation}

(2.14)

Furthermore, the function \( \psi(u) \) defines a strictly increasing involution such that \( \psi^2(u) = u \), which maps the interval \([-\infty,c]\) to the interval \([c,\infty)\).

**Proof.** By Lemma 2.2 the condition (2.6) is required. By Lemma 2.1 (solution of AMP(3)) the atoms are the distinct real zeros of the standard cubic orthogonal polynomial of degree three \( p_3(x) \), which satisfies the three linear expected value equations

\[ E[X'p_3(X)] = 0, \quad i = 0,1,2. \]

(2.15)

The condition

\[ E[p_3(X)] = E[(X - x_1)(X - x_2)(X - x_3)] = \gamma - (x_1 + x_2 + x_3) - x_1x_2x_3 = 0 \]

implies the relationship \( x_2 = \varphi(x_1,x_3) \). Inserted into the condition

\[ E[Xp_3(X)] = E[X(X - x_1)(X - x_2)(X - x_3)] = \gamma_2 + 3 - (x_1 + x_2 + x_3)\gamma + (x_1x_2 + x_1x_3 + x_2x_3) = 0, \]

one obtains that \( x_3 \) is solution of the quadratic equation \( q(x_1)x_3^2 - A(x_1)x_3 - B(x_1) = 0 \), hence \( x_3 = \psi(x_1) \) as defined in (2.9). The probabilities take the values

\[ p_1 = \frac{1 + x_1x_3}{(x_2 - x_1)(x_3 - x_1)}, \quad p_2 = \frac{-(1 + x_1x_3)}{(x_2 - x_1)(x_3 - x_2)}, \quad p_3 = \frac{1 + x_1x_2}{(x_1 - x_1)(x_3 - x_2)}. \]

(2.16)

Since \( x_1 < x_2 < x_3 \) one must have \( 1 + x_1x_3 \geq 0, \quad 1 + x_1x_2 \leq 0, \quad 1 + x_2x_3 \geq 0 \). Since \( x_1x_3 \leq -1 \) one must have \( x_1 < 0 < x_3 \), hence also \( \overline{x}_1 < 0 < \overline{x}_3 \). It follows that

\[ \overline{x}_1 \cdot (1 + x_1x_3) = \overline{x}_3 - x_2 \leq 0, \]

\[ \overline{x}_3 \cdot (1 + x_1x_3) = \overline{x}_3 - x_1 \geq 0, \]

\[ \overline{x}_1 \cdot (1 + x_2x_3) = \overline{x}_2 - x_2 \geq 0, \]

which implies the inequalities \( x_1 \leq \overline{x}_1 < 0, \quad \overline{x}_3 \leq x_2 \leq \overline{x}_1 \). Since \( x_2 = \varphi(x_1,x_3) \) the second inequalities in (2.19) are equivalent with \( x_2^2 - px_2 - 1 \geq 0, \quad x_1^2 - px_1 - 1 \geq 0 \). Since \( x_1 < 0 < x_3 \) one must have \( (x_1,x_3) \in (-\infty,c] \times [c,\infty) \) and \( x_2 \in [c,\overline{c}] \). This shows (2.9). To obtain (2.10), note that \( x_2 = \varphi(x_1,\psi(x_1)) \) and \( x_3 = \psi(x_1) \) are solutions of the quadratic equation \( q(x_1)x^2 - A(x_1)x - B(x_1) = 0 \). One calculates
\[
(x_2 - x_i)(x_3 - x_i) = x_2^2 - (x_2 + x_3)x_i + x_2x_3 = x_i^2 - \frac{A(x_i)}{q(x_i)}x_i - \frac{B(x_i)}{q(x_i)} = -\frac{q(x_i)^2 + \Delta(1 + x_i^2)}{q(x_i)},
\]

\[1 + x_2x_3 = -\frac{\Delta}{q(x_i)}.\] Inserted into (2.16) one gets \(p_i = p(x_i)\). Making cyclic permutations of \(x_1, x_2, x_3\) one obtains \(p_i = p(x_i), \quad i = 2,3\).


Based on the preceding two Sections, we give an explicit proof of how to construct the stochastic ordered maximal random variables in the situations applied in the present work.

**Theorem 3.1.** The distribution function of the standardized (Chebyshev-Markov) stochastic ordered maximal random variable \(X_{\mu,\text{max}}^{(2)}\) for the set \(D_2 := D_2([a,b], \mu = 0, \sigma = 1)\) is described in Table 3.1.

**Table 3.1 :** Stochastic ordered maximal standard distribution for the set \(D_2\)

<table>
<thead>
<tr>
<th>condition</th>
<th>(F_{\text{str},\text{max}}^{(2)}(x))</th>
<th>extremal support of (Z_\delta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a \leq x \leq \bar{b})</td>
<td>0</td>
<td>({x, \bar{x}})</td>
</tr>
<tr>
<td>(\bar{b} \leq x \leq \bar{a})</td>
<td>(\frac{1 + bx}{(b-a)(x-a)})</td>
<td>({a, x, b})</td>
</tr>
<tr>
<td>(\bar{a} \leq x &lt; b)</td>
<td>(\frac{x^2}{1 + x^2})</td>
<td>({\bar{x}, x})</td>
</tr>
<tr>
<td>(x = b)</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** By Table 1.1, quadratic polynomial majorants \(q_\delta(z)\) for the Heaviside indicator function \(I_{[x,\infty]}(z)\) are obtained either for ordered biatomic supports \(\{x, \bar{x}\}, \{\bar{x}, x\}\) (case (4)) or for a triatomic support \(\{a, x, b\}\) (case (1)). With Proposition 2.1, one has necessarily \(x \in [a, \bar{b}]\) if \(\{x, \bar{x}\}\) is the extremal support, and \(x \in [\bar{a}, b]\) if \(\{\bar{x}, x\}\) is the extremal support. Similarly, the triatomic support \(\{a, x, b\}\) is only feasible if \(x \in [\bar{b}, \bar{a}]\). These supports define standard bi- and triatomic random variables \(Z_\delta\) such that \(\Pr(q_\delta(Z_\delta) = I_{[x,\infty]}(Z_\delta)) = 1\). The displayed extremal values follow from the calculation of \(F_{\text{str},\text{max}}^{(2)}(x) = 1 - E[I_{[x,\infty]}(Z_\delta)]\) for each of the three cases.

**Theorem 3.2.** The distribution function of the standardized (Chebyshev-Markov) stochastic ordered maximal random variable \(X_{\gamma,\text{max}}^{(4)}\) for the set \(D_4 := D_4((-\infty, \infty), \mu = 0, \sigma = 1, \gamma_1, \gamma_2)\) is described in Table 3.2 with the notations of Proposition 2.2.
Table 3.2: Stochastic ordered maximal standard distribution for the set $D_4$

<table>
<thead>
<tr>
<th>condition</th>
<th>$F_{st,\text{max}}^{(4)}(x)$</th>
<th>extremal support of $Z_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \leq c$</td>
<td>0</td>
<td>${x, \varphi(x), \psi(x), \psi(x)}$</td>
</tr>
<tr>
<td>$c \leq x \leq \bar{c}$</td>
<td>$p(\psi(x))$</td>
<td>${\psi(x), x, \varphi(x), \psi(x)}$</td>
</tr>
<tr>
<td>$x \geq \bar{c}$</td>
<td>$1 - p(x)$</td>
<td>${\varphi(x), \psi(x), \psi(x), x}$</td>
</tr>
</tbody>
</table>

**Proof.** By Table 1.1, case (4), and Proposition 2.2, one observes that biquadratic polynomial majorants $q_x(z) \geq I_{[x,\infty)}(z)$ can only be obtained at the ordered extremal supports displayed in Table 3.2. These supports define standard triatomic random variables $Z_x$ such that $\Pr(q_x(Z_x) = I_{[x,\infty)}(Z_x)) = 1$. The displayed extremal values follow from the calculation of $F_{st,\text{max}}^{(4)}(x) = 1 - E[I_{[x,\infty)}(Z_x)]$ for each of the three cases. ◊

4. Stop-loss ordered bounds.

The present Section contains an explicit proof of how to construct the stop-loss ordered maximal random variables in the situations applied in the present work.

**Theorem 4.1.** The maximal stop-loss transform $\pi^{(2)}_{\text{max}}(d) = \max_{x \in D_2} \{E[(X - d)_+], \}$ for the set $D_2 := D_2([a,b], \mu = 0, \sigma = 1)$ and the distribution function $F_{st,\text{max}}^{(2)}(x)$ of the corresponding standardized stop-loss ordered maximal random variable $X_{st,\text{max}}^{(2)}$ are described in Table 4.1.

Table 4.1: Maximal stop-loss transform and stop-loss ordered maximal distribution for $D_2$

<table>
<thead>
<tr>
<th>condition</th>
<th>$F_{st,\text{max}}^{(2)}(d)$</th>
<th>$\pi^{(2)}_{\text{max}}(d)$</th>
<th>extremal support</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \leq d \leq \frac{1}{2}(a + \bar{a})$</td>
<td>$\frac{1}{1 + a^2}$</td>
<td>$\frac{1}{1 + a^2}$</td>
<td>${a, \bar{a}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}(a + \bar{a}) \leq d \leq \frac{1}{2}(b + \bar{b})$</td>
<td>$\frac{1 + d}{\sqrt{1 + d^2}}$</td>
<td>$\frac{1}{2}\left(\sqrt{1 + d^2} - d\right)$</td>
<td>${d - \sqrt{1 + d^2}, d + \sqrt{1 + d^2}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}(b + \bar{b}) \leq d &lt; b$</td>
<td>$\frac{b^2}{1 + b^2}$</td>
<td>$\frac{b - d}{1 + b^2}$</td>
<td>${\bar{b}, b}$</td>
</tr>
<tr>
<td>$d = b$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** The distribution function $F_x(x)$ and the stop-loss transform $\pi_x(x)$ of a random variable $X$ are related by the relationship $F_x(x) = 1 + \pi_x(x)$. Therefore, it suffices to determine the maximal stop-loss transform. By Table 1.2 quadratic polynomial majorants $q(X) \geq (X - d)_+$, $d$ the deductible of the stop-loss function, can only be obtained at diatomic supports of the forms $\{a, \bar{a}\}$ (case (3a)), $\{x, \bar{x}\}$ (case (4a)) or $\{\bar{b}, b\}$ (case (2a)).

Case (1): extremal support $\{a, \bar{a}\}$
The unique quadratic polynomial \( q(X) = q(X; a, \bar{a}, d) \) such that \( q(a) = 0, q(\bar{a}) = \bar{a} - d, q'(\bar{a}) = 1 \), is given by
\[
q(X) = \frac{(d - a)(X - \bar{a})^2}{(\bar{a} - a)^2} + (X - d).
\]

Solving the condition \( Q_1(x; a, \bar{a}, d) = q(x; a, \bar{a}, d) = 0, x \leq a \), one finds for the deductible
\[
d = \frac{\bar{a}^3 - ax}{2\bar{a} - a - x}, \quad x \leq a,
\]
which implies the desired results (note that for \( x \to -\infty \) one has \( d \to a \) and for \( x = a \) one has \( d = \frac{1}{2}(a + \bar{a}) \)).

**Case (2) : extremal support \( \{x, \bar{x}\} \)**

By Proposition II.1 the ordered diatomic support \( \{x, \bar{x}\} \) is feasible exactly when \( x \in [a, \bar{b}] \).

The unique quadratic polynomial \( q(X) = q(X; x, \bar{x}, d) \) such that \( q(x) = q'(x) = 0, q(\bar{x}) = \bar{x} - d \), is given by
\[
q(X) = \frac{(\bar{x} - d)(X - x)^2}{(\bar{x} - x)^2}.
\]

Solving the condition \( Q_2(x; x, \bar{x}, d) = q'(\bar{x}) - 1 = 0 \), one finds
\[
d = \frac{1}{2}(x + \bar{x}), \quad a \leq x \leq \bar{b},
\]
from which all statements follow.

**Case (3) : extremal support \( \{\bar{b}, b\} \)**

The unique quadratic polynomial \( q(X) = q(X; \bar{b}, b, d) \) such that \( q(\bar{b}) = q'(\bar{b}) = 0, q(b) = b - d \), is given by
\[
q(X) = \frac{(b - d)(X - \bar{b})^2}{(b - \bar{b})^2}.
\]

Solving the condition \( Q_2(x; \bar{b}, b, d) = q(x) - (x - d) = 0, x \geq b \), one finds
\[
d = \frac{bx - \bar{b}^2}{x + b - 2\bar{b}}, \quad x \geq b,
\]
and the stated results are immediately checked. \( \diamond \)
Theorem 4.2. The maximal stop-loss transform \( \pi_{\text{max}}^{(4)}(d) = \max_{X \in D_4} \{ E[(X - d)^+ |] \} \) for the set \( D_4 \) and the distribution function \( F_{\text{sl, max}}^{(4)}(d) \) of the corresponding standardized stop-loss ordered maximal random variable \( X_{\text{sl, max}}^{(4)} \) are implicitly described in Table 4.2 using the “deductible” function

\[
d(x) = \frac{1}{2} \left\{ \phi(x, \psi(x)) - x \right\} + x - \psi(x) - d(x)
\]

and the notations introduced in Proposition 4.2.

<table>
<thead>
<tr>
<th>condition</th>
<th>( F_{\text{sl, max}}^{(4)}(d(x)) )</th>
<th>( \pi_{\text{max}}^{(4)}(d(x)) )</th>
<th>extremal support of ( Z_{d(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \leq c )</td>
<td>( p(x) )</td>
<td>( p(x) \cdot (d(x) - x) - d(x) )</td>
<td>( { x, \phi(x, \psi(x)), \psi(x) } )</td>
</tr>
<tr>
<td>( x \geq \bar{c} )</td>
<td>( 1 - p(x) )</td>
<td>( p(x) \cdot (-d(x)) )</td>
<td>( { \phi(x, \psi(x)), \psi(x), \bar{x} } )</td>
</tr>
</tbody>
</table>

Proof. Applying the chain rule of differential calculus to the identity \( F_X(x) = 1 + \pi_X^\prime(x) \) between the distribution function and stop-loss transform of a random variable \( X \), one obtains the relationship

\[
F_X(d(x)) = 1 + \frac{(\pi \circ d)'(x)}{d'(x)}. \tag{4.2}
\]

Therefore, it suffices to determine the maximal stop-loss transform. From Table 1.2, case (4a), and Proposition 2.2, it follows that biquadratic polynomial majorants \( q(X) \geq (X - d)_+ \) can only be constructed for supports containing the three atoms \( x, \phi(x, \psi(x)), \psi(x) \). Set \( f(x) = (x - d)_+ \) and let us partition the interval \( I = (-\infty, \infty) \) in the two pieces \( I_1 = (-\infty, d] \), \( I_2 = [d, \infty) \) such that \( f(x) = \ell_1(x) = 0 \) on \( I_1 \), \( f(x) = \ell_2(x) = x - d \) on \( I_2 \). The piecewise biquadratic function \( Q(x) = q(x) - f(x) \) coincides on \( I_i \) with the polynomial \( Q_i(x) = q(x) - \ell_i(x), \quad i = 1, 2 \). Furthermore, let \( u, v < w \) represent zeros of \( Q_i(x), \quad i = 1, 2 \). Two cases can occur.

Case (1) : \( u \in I_1 \) is a double zero of \( Q_1(x) \), \( v, w \in I_2 \) are double zeros of \( Q_2(x) \) as in the following figure:

```
    a----------u----------d----------v----------w
```

Suppose first that \( u \) is a simple zero. Since a biquadratic polynomial is uniquely determined by five conditions, there exists a unique \( q(x) \) with the required conditions, namely
\[ q(x) = \frac{\ell_1(u) - \ell_2(u)}{(v-u)^2(w-u)^2} (x-v)^2(x-w)^2 + \ell_2(x). \]

Solving the additional condition \( Q'_2(u) = 0 \) for a double zero, one obtains
\[
d = \frac{1}{2} \left\{ \frac{v w + u (v + w) - 3 u^2}{v + w - 2 u} \right\} = \frac{1}{2} \left( \frac{(v-u)(u+w) + 2(w-u)u}{(v-w) + (w-u)} - u \right). \]

The extremal support \( \{u,v,w\} = \{x,\varphi(x),\psi(x)\} \), feasible by (2.9) provided \( x \leq c \), yields the formula for \( d(x) \) as well as the maximal stop-loss transform
\[
\pi_{\text{max}}(d) = p(v) \cdot (v-d) + p(w) \cdot (w-d) = -p(u)u - (1-p(u))d = p(u) \cdot (d-u) - d.
\]

Case (2): \( u, v \in I_1 \) are double zeros of \( Q_1(x) \), \( w \in I_2 \) is a double zero of \( Q_2(x) \) as in the following figure:

Suppose first that \( w \) is a simple zero. By symmetry to case (1), the unique biquadratic polynomial with the required conditions is
\[ q(x) = \frac{\ell_2(w) - \ell_1(w)}{(w-u)^2(w-v)^2} (x-u)^2(x-v)^2 + \ell_1(x). \]

The additional condition \( Q'_1(w) = 0 \) for a double zero yields
\[
d = \frac{1}{2} \left( \frac{(w-v)(u+w) + 2(w-u)w}{(v-w) + (w-u)} - u \right). \]

The extremal support \( \{u,v,w\} = \{\varphi(x),\psi(x),\varphi(x),x\} \), which by (2.9) and the involution property of \( \psi(x) \) is feasible provided \( x \geq \overline{c} \), yields the formula for \( d(x) \) as well as the maximal stop-loss transform \( \pi_{\text{max}}(d) = p(w) \cdot (w-d) \).

Finally, it remains to be shown that \( (\pi \circ d)(x) \) is a well-defined function. In case (1) and for the limiting case \( x \to c \) the triatomic extremal support \( \{x,\varphi(x),\psi(x),x\} \) converges to the biatomic support \( \{c,\overline{c}\} \) and \( d(x) \) converges to \( \frac{1}{2} \gamma \). By symmetry, the same holds in case (2) for the limiting case \( x \to \overline{c} \). Furthermore \( d(x) \) is strictly increasing. Therefore, in case (1) the function \( d(x) \) maps \( (-\infty, c] \) one-to-one on \( (-\infty, \frac{1}{2} \gamma] \), and in case (2) it maps \( \overline{c}, \infty \) one-to-one on \( [\frac{1}{2} \gamma, \infty) \). It follows that \( (\pi \circ d)(x) \) is a well-defined function. \( \diamond \)
APPENDIX D

Covariance formulas for the linear Spearman copula

Starting point is the following result.

**Theorem D.1.** Let \((X,Y)\) be distributed as \(F(x,y) = C_\theta[F_X(x),F_Y(y)]\), where \(C_\theta(u,v)\) is the linear Spearman copula (VII.3.1). Assume that the continuous and strictly increasing marginal distributions are defined on the open supports \((a_X,b_X)\), \((a_Y,b_Y)\). For an arbitrary differentiable function \(\psi(y)\), assume the following regularity assumption holds:

\[
\lim_{y \to a_Y} \psi(y) F_Y(y) = 0, \\
\lim_{y \to b_Y} \psi(y) E[X] F_Y(y) - \int_{a_Y}^{b_Y} F_X^{-1}[F_Y^\theta(y)] dF_Y(y) = 0. 
\]

Then one has the covariance formula

\[
\text{Cov}[X,\psi(Y)] = \text{sgn}(\theta) \cdot E[F_X^{-1}[F_Y^\theta(Y)] - E[X]] \cdot \psi(Y), \tag{D.1}
\]

where one sets

\[
F_Y^\theta(y) = \begin{cases} 
F_Y(y), & \theta \geq 0, \\
F_Y^\theta(y), & \theta < 0,
\end{cases} \tag{D.2}
\]

with \(F_Y^\theta(y) = 1 - F_Y(y)\) the survival function.

**Proof.** Let us first derive the regression function \(E[X|Y = y]\). The conditional distribution of \(X\) given \(Y = y\) equals for \(\theta \geq 0\)

\[
F(x|y) = \frac{\partial C_\theta}{\partial v}[F_X(x),F_Y(y)] = \left\{ \begin{array}{ll}
F_X(x) + \theta F_X^\theta(x), & x \geq F_X^{-1}[F_Y(y)] \\
(1 - \theta) F_X(x), & x < F_X^{-1}[F_Y(y)]
\end{array} \right. \tag{D.3}
\]

and for \(\theta < 0\)

\[
F(x|y) = \left\{ \begin{array}{ll}
(1 + \theta) F_X(x), & x < F_X^{-1}[F_Y(y)] \\
F_X(x) - \theta F_X^\theta(x), & x \geq F_X^{-1}[F_Y(y)]
\end{array} \right. \tag{D.4}
\]

Through calculation one obtains the regression formula

\[
E[X|y] = \int_0^\infty [1 - F(x|y)] dx - \int_0^y F(x|y) dx \
= \left\{ \begin{array}{ll}
E[X] - \theta \cdot (E[X] - F_X^{-1}[F_Y(y)]), & \theta \geq 0, \\
E[X] + \theta \cdot (E[X] - F_X^{-1}[F_Y(y)]), & \theta \leq 0,
\end{array} \right. \tag{D.5}
\]
which is a weighted average of the mean $E[X]$ and the quantile $F_X^{-1}[F_y(y)]$ respectively $X E$ and the quantile $Y X (1 - F(x) \leq y) dx - \int_0^1 F_X(x) dx$. To obtain the stated covariance formula, one uses the well-known formula by Hoeffding(1940) and Lehmann(1966), Lemma 2, to get the expression

\[
Cov[X, \psi(Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x, y) - F_X(x)F_Y(y)] \psi'(y) dy dx
\]

\[
= \int_{-\infty}^{\infty} F_Y(y) \left[ F(x \mid y \leq y) - F_X(x) \right] \psi'(y) dy dx
\]

\[
= \int_{-\infty}^{\infty} \left( E[X] - E[X \mid y \leq y] \right) F_Y(y) \psi'(y) dy.
\]

Furthermore, from (D.6) one obtains

\[
E[X] - E[X \mid y \leq y] = sgn(\theta) \theta \cdot (E[X] - E[F_X^{-1}[F_Y^\theta(Y)] \mid y \leq y]).
\]

(D.8)

Inserted in (D.7), a partial integration yields

\[
Cov[X, \psi(Y)] = sgn(\theta) \theta \cdot \int_{a_y}^{b_y} \left( E[X] F_Y(y) - \int_{a_y}^{b_y} F_X^{-1}[F_Y^\theta(y)] dF_Y(y) \right) \psi'(y) dy
\]

\[
= sgn(\theta) \theta \cdot \left( \psi(y) \left[ E[X] F_Y(y) - \int_{a_y}^{b_y} F_X^{-1}[F_Y^\theta(y)] dF_Y(y) \right]_{a_y}^{b_y} \right.
\]

\[
\left. + \int_{a_y}^{b_y} \psi(y) \left( F_X^{-1}[F_Y^\theta(y)] - E[X] \right) dF_Y(y) \right).
\]

(D.9)

which implies (D.1) by the regularity assumption. ◊

The application of this result to margins from a symmetric location-scale family is simple.

**Corollary D.1.** Under the assumptions from Theorem D.1 suppose that

\[ F_X(x) = F_Z \left( \frac{x - \mu_X}{c_X} \right), \quad F_Y(x) = F_Z \left( \frac{x - \mu_Y}{c_Y} \right), \quad \text{with} \quad \mu_X = E[X], \mu_Y = E[Y], \quad \text{and} \]

\[ F_Z(-z) = F_Z(z). \] Then one has

\[ Cov[X, \psi(Y)] = \theta \frac{c_X}{c_Y} Cov[Y, \psi(Y)]. \]

(D.10)

**Proof.** The result follows from (D.1) noting that \( F_X^{-1}[F_Y^\theta(y)] = \mu_X + sgn(\theta) \frac{c_X}{c_Y} (y - \mu_Y). \) ◊

**Remark D.1.** As shown in Section 3.1 one has \( \theta = \rho_s \) the Spearman grade correlation coefficient. If \( \psi(y) = y \) satisfies (RA), and the variances \( \sigma_X^2, \sigma_Y^2 \) exist, one has for a symmetric location-scale family
\[ \rho_s \cdot \frac{c_x}{c_y} = \rho \cdot \frac{\sigma_x}{\sigma_y}, \]  

\[(D.11)\]

where \( \rho \) is Pearson’s correlation coefficient. If the scale parameters are proportional to the standard deviations, that is \( c_x = c \cdot \sigma_x \) and \( c_y = c \cdot \sigma_y \) for some \( c \), then \( \rho_s = \rho \). This special case shows why Spearman’s \( \rho_s \) should be interpreted as a measure of correlation. Further special cases under which \( \rho_s = \rho \) will be derived later in this Appendix.

In general, if \( \psi(y) = y \) satisfies (RA), the parameter \( \theta = \rho_s \) remains non-linearly related with Pearson’s correlation coefficient \( \rho \), as the examples below illustrate. If one has to fit the model to data, the estimation of \( \theta = \rho_s \) and \( \rho \) is a very important issue. In case \( \theta > 0 \) the model captures a non-trivial asymptotic dependence by (VII.3.10), and is thus capable to model in some way bivariate extreme values. In the latter situation, the standard product-moment correlation estimator for \( \rho \) has a very bad performance, and there is a need for more robust estimators (e.g. Lindskog(2000)). In this respect the relationship (D.1) opens the way for alternative estimators. For example, if one follows the method of inference function for margins or IFM method studied in McLeish and Small(1988), Xu(1996), and Joe(1997), Section 10.1, one proceeds by doing an estimation of the parameters from the univariate marginal distributions (through separate maximum likelihood estimations) followed by an estimation of the dependence parameter \( \theta \). Inserting the estimated parameters in an explicit analytical expression for (D.1) yields an alternative estimator of the linear correlation coefficient by setting

\[ \hat{\rho} = \text{sgn}(\hat{\theta}) \cdot \frac{E[Q_x \cdot F_y(\hat{Y})] - E[X] \cdot \hat{Y}}{\hat{\sigma}_x \cdot \hat{\sigma}_y}, \]  

\[(D.12)\]

where the hat indicates insertion of estimated parameters. A new simple estimator for the dependence parameter \( \theta = \rho_s \) is provided in the following result.

**Theorem D.2.** Let \( \{(X_{k},Y_{k})\} \) be a sample of size \( N \) from \( LS_{\theta}(X,Y) \), \( \theta \in [-1,1] \), and let \( X_{(k)}, Y_{(k)} \) denote the order statistics taken in the decreasing order. Denote by \( iX_{(k)} \) the inverse rank of the observation \( X_{(k)} \), defined as the index value \( j \) in the original sample such that the rank of \( X_j \) in the order statistics is \( k \). Then the formulas

\[ |\hat{\theta}| = \left| \frac{\sum_{k=1}^{N}(X_k - \bar{X}) \cdot (Y_k - \bar{Y})}{\sum_{k=1}^{N}(X_{iX_{(k)}} - \bar{X}) \cdot (Y_{iX_{(k)}} - \bar{Y})} \right|, \]  

\[(D.13)\]

\[ \text{sgn}(\hat{\theta}) = \text{sgn}\left( \sum_{k=1}^{N}(X_k - \bar{X}) \cdot (Y_k - \bar{Y}) \right), \]  

\[(D.14)\]

yield an estimator of Spearman’s coefficient \( \theta = \rho_s \).
Proof. If $\psi(y) = y$ satisfies (RA), one knows by Theorem D.1 that
\[
\text{sgn}(\theta)\theta = \frac{\text{Cov}[X,Y]}{E[(Q_XF^\theta(Y)) - E[X]Y]},
\]
An estimator of the numerator is the usual product moment estimator. For the denominator use that $F^\theta(Y)$ is uniformly distributed on $(0,1)$ to get the symmetric expression
\[
E[(Q_XF^\theta(Y)) - E[X]Y] = E[(Q_X(U) - E[X])Q_Y(U)] = E[(Q_X(U) - E[X])(Q_Y(U) - E[Y])].
\]
To obtain an estimator for this quantity, let us use the empirical quantile functions $Q_X^u(u), Q_Y^u(u)$ of the samples. Using Embrechts et al. (1997), p.183, one has the order statistics representation $Q_X(u) = X_{(k)}$ provided $1 - \frac{k}{n} \leq u < 1 - \frac{k-1}{n}$. Equivalently, in terms of the inverse rank function, one has $Q_X^u(u) = X_{irX_{[n]}}$, $Q_Y^u(u) = Y_{irY_{[n]}}$, where $[x]$ denotes the integer part of $x$. The desired estimator is the empirical counterpart of the above symmetric expression. \diamond

In the remaining part of this Appendix, several examples illustrate the analytical evaluation of (D.1), which provide explicit relationships between the parameters $\theta = \rho_s$ and $\rho$.

Example D.1: lognormal margins

Suppose that $F_X(x) = D\left(\frac{\exp(-\alpha)}, \frac{\exp(-\beta)}{\beta_s}\right)$, $F_Y(y) = D\left(\frac{\exp(-\alpha)}, \frac{\exp(-\beta)}{\beta_t}\right)$, with $D(D(z) = D(z)$ (note that if $\theta = \rho_s \geq 0$ the function $D(z)$ may be non-symmetric). Then one has
\[
Q_X\left[Q_Y^\theta(y)\right] = \exp[\alpha_\theta \cdot \ln(y) + \beta_\theta], \quad \alpha_\theta = \text{sgn}(\theta) \cdot \frac{\beta_s}{\beta_t}, \quad \beta_\theta = \alpha_x - \alpha_\theta \alpha_y,
\]
and under the regularity assumption (RA), one has
\[
\text{Cov}[X,\psi(Y)] = \text{sgn}(\theta)\theta \cdot \text{E}\left[\exp(\beta_\theta \cdot Y^{-\alpha_\theta} - E[X]) \cdot \psi(Y)\right]. \quad (D.15)
\]
In the special case $D(z) = \Phi(z)$ of lognormal margins, and $\psi(y) = y$, standard calculations show that (RA) is fulfilled and one obtains
\[
\text{Cov}[X,Y] = \text{sgn}(\theta)\theta \cdot \mu_X \mu_Y \cdot \left(e^{\text{sgn}(\theta) \beta_s \beta_t} - 1\right), \quad (D.16)
\]
where $\mu_X = \exp\left[\alpha_x + \frac{1}{2} \beta_x^2\right], \quad \mu_Y = \exp\left[\alpha_y + \frac{1}{2} \beta_y^2\right]$ denote the means of $X$ and $Y$. In the special case of equal coefficients of variation $k_X = k_Y = \sqrt{\exp(\beta_x^2) - 1} = \sqrt{\exp(\beta_y^2) - 1}$ and $\theta \geq 0$, one has
\[
\text{Cov}[X,Y] = \theta \cdot \sigma_X \sigma_Y, \quad (D.17)
\]
where $\sigma_X, \sigma_Y$ are the standard deviations. In this special case $\theta = \rho_s$ identifies with Pearson’s $\rho$. In general one has the relationship
\[ \rho = \frac{\text{sgn}(\rho_x) \rho_s \cdot \left( e^{\text{sgn}(\theta) \sigma_x \sigma_y} - 1 \right)}{k_x k_y}. \]  

(D.18)

In particular, the extremal bounds for \( \rho \), which are attained when \( \rho_s = \pm 1 \) (e.g. Tchen(1980)), are given by

\[ \frac{e^{-\sigma_x \sigma_y} - 1}{k_x k_y} \leq \rho \leq \frac{e^{\sigma_x \sigma_y} - 1}{k_x k_y}. \]  

(D.19)

This has been originally derived by DeVeaux(1976) as mentioned in Romano and Siegel(1986), Section 4.22. As a significant example, if \( \beta_x = 1, \beta_y = 4 \), then \( \rho \) must be close to zero. This illustrates the fact that by fixed \( \rho \) bivariate distributions with lognormal margins do not always exist. In contrast, by fixed \( \theta = \rho_s \), the linear Spearman copula guarantees the existence of a bivariate distribution with lognormal margins.

**Example D.2**: gamma margins

Suppose that \( F_x(x) = \Gamma(\beta_x, x, \alpha_x) \), \( F_y(y) = \Gamma(\beta_y, y, \alpha_y) \), where \( \Gamma(\cdot, \cdot) \) is the Gamma distribution with shape parameter \( \alpha \). Restricting the attention to \( \theta \geq 0 \), one obtains from

\[ Q_x[F_y(y)] = \beta_y^{-1}(\Gamma(\beta_y, y, \alpha_y), \alpha_x) \]  

and (D.1) the formula

\[ \rho = \frac{\theta}{\sigma_x \sigma_y} \left[ \frac{1}{\beta_x} \int_0^\infty \Gamma^{-1}(\Gamma(\beta_y, y, \alpha_y), \alpha_x) y dF_y(y) - \mu_x \mu_y \right] \]  

(D.21)

If \( \alpha_x = \alpha_y \) (equal coefficients of variation) it is not difficult to show that Spearman’s \( \theta = \rho_s \) coincides with Pearson’s \( \rho \).

**Example D.3**: Pareto margins

If \( X \sim \text{Par}(\lambda_x, \gamma_x), Y \sim \text{Par}(\lambda_y, \gamma_y) \) have Pareto survival functions \( F_x(x) = (1 + \frac{x}{\lambda_x})^{-\gamma_x}, \)

\[ F_y(y) = \left( 1 + \frac{y}{\lambda_y} \right)^{-\gamma_y}, \quad x \geq 0, \quad \lambda_x, \lambda_y > 0, \quad \gamma_x, \gamma_y > 1, \] 

one obtains with the quantile function

\[ Q_x[u] = \lambda_x \cdot \left[ (1 - u)^{-\gamma_x} - 1 \right] \] 

that

\[ \text{Cov}[X, Y] = \theta \cdot E \left[ \lambda_x \left( 1 + \frac{Y}{\lambda_y} \right)^{-\gamma_y} \cdot Y - \lambda_x Y - \mu_x Y \right] \]  

(D.22)

Through straightforward calculation one gets

\[ I = E \left[ \lambda_x \left( 1 + \frac{Y}{\lambda_y} \right)^{-\gamma_y} \cdot Y \right] = \lambda_x \left( \frac{\gamma_y}{\gamma_x - 1} \right) \int_0^\infty \frac{y \left( \frac{\gamma_y}{\gamma_x} - 1 \right)}{\gamma_y \lambda_y} \left( 1 + \frac{y}{\lambda_y} \right)^{-\gamma_y + 1} dy \]  

(D.23)
Since the last integral represents the mean of a random variable, which is \( \text{Par} \left( \lambda, \gamma X Y - 1 \right) \) distributed, one obtains

\[
I = \lambda X \left( \frac{Y X}{Y X - 1} \right) \lambda Y \left( \frac{Y X}{Y X Y - Y X - Y Y} \right), \quad Y X Y > Y X + Y Y. \tag{D.24}
\]

Inserted in (D.22) one obtains the covariance formula

\[
\text{Cov}[X, Y] = \theta \left( \frac{Y X Y}{Y X Y - Y X - Y Y} \right) \mu X \mu Y. \tag{D.25}
\]

In the special case of equal coefficients of variation \( k = \frac{Y X}{Y X - 2} = \frac{Y Y}{Y Y - 2}, \quad Y X = Y Y > 2, \)

one has \( \text{Cov}[X, Y] = \theta \cdot \sigma X \sigma Y, \) hence \( \theta = \rho s \) coincides with Pearson’s \( \rho \) as in the Examples D.1 and D.2.

**Example D.4**: log-double Weibull margins

Let \( X \sim \text{ln} \text{DW}(\alpha X, \beta X, \gamma X), \quad Y \sim \text{ln} \text{DW}(\alpha Y, \beta Y, \gamma Y) \) have log-double Weibull distribution functions \( F X (x) = F Y X \left( \ln(x) - \alpha X \right) \beta X, \quad F Y (x) = F Y Y \left( \ln(x) - \alpha Y \right) \beta Y, \) with

\[
F Y X (z) = \begin{cases} \frac{1}{2} \exp \left( - \lambda X |z|^\gamma X \right), & z \leq 0, \\ 1 - \frac{1}{2} \exp \left( - \lambda X |z|^\gamma X \right), & z \geq 0, \end{cases} \quad \lambda X = \Gamma \left( 1 + \frac{2}{Y X} \right), \tag{D.26}
\]

\[
F Y Y (z) = \begin{cases} \frac{1}{2} \exp \left( - \lambda Y |z|^\gamma Y \right), & z \leq 0, \\ 1 - \frac{1}{2} \exp \left( - \lambda Y |z|^\gamma Y \right), & z \geq 0, \end{cases} \quad \lambda Y = \Gamma \left( 1 + \frac{2}{Y Y} \right). \tag{D.27}
\]

A calculation shows that

\[
Q X [F Y (y)] = \exp \left( \alpha X + \beta X \left( \frac{\lambda Y}{\lambda X} \right) \gamma Y \frac{\ln(y) - \alpha Y}{\beta Y} \right) \frac{\ln(y) - \alpha Y}{\beta Y} \gamma Y. \tag{D.28}
\]

Let \( f Y (x) = \frac{1}{\beta X} f Y X \left( \ln(x) - \alpha X \right) \beta X \) the density of \( Y, \) where \( f Y X (z) = \frac{1}{2} \gamma X \lambda X |z|^\gamma X \frac{\ln(y) - \alpha Y}{\beta Y} \gamma Y \exp \left( - \lambda Y |z|^\gamma Y \right). \) With the substitution \( \ln(y) - \alpha Y = \frac{\ln(y) - \alpha Y}{\beta Y} \beta Y z \) one obtains

\[
I = E \left[ Q X [F Y (y)] \right] = e^{\alpha Y + \alpha Y} \cdot (I_1 + I_2), \tag{D.29}
\]

\[
I_1 = \int_0^\infty \exp \left( - \beta X \left( \frac{\lambda Y}{\lambda X} \right) \gamma X - \beta Y z \right) \frac{1}{2} \gamma Y \lambda Y z^\gamma Y \exp \left( - \lambda Y z^\gamma Y \right) dz, \tag{D.30}
\]
\[ I_1 = \int_0^\infty \exp \left\{ \beta_x \left( \frac{u}{\lambda_x} \right)^{\gamma_x} + \beta_y z \right\} \frac{1}{2} \gamma_x \lambda_x z^{\gamma_x - 1} \exp \left\{ -\lambda_y z^{\gamma_y} \right\} dz. \]  

(D.31)

The further substitution \( u = \lambda_x z^{\gamma_x} \) yields

\[ I_1 = \int_0^\infty \exp \left\{ -\beta_x \left( \frac{u}{\lambda_x} \right)^{\gamma_x} - \beta_y \left( \frac{u}{\lambda_y} \right)^{\gamma_y} \right\} e^{-u} du. \]  

(D.32)

Expanding the first exponential expression in a Taylor series and using the integral definition of the Gamma function, one sees that

\[ I_1 = \frac{1}{2} \sum_{k=0}^\infty \left( \frac{-1}{k!} \right)^k \left[ \frac{\beta_x}{\lambda_x^{\gamma_x}} \right]^k \left[ \frac{\beta_y}{\lambda_y^{\gamma_y}} \right]^{k-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{k-j}{\gamma_y} \right). \]  

(D.33)

Similarly, one obtains

\[ I_2 = \frac{1}{2} \sum_{k=0}^\infty \left( \frac{-1}{k!} \right)^k \left[ \frac{\beta_x}{\lambda_x^{\gamma_x}} \right]^k \left[ \frac{\beta_y}{\lambda_y^{\gamma_y}} \right]^{k-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{k-j}{\gamma_y} \right). \]  

(D.34)

Inserting (D.33) and (D.34) into (D.29) and using (D.1) one obtains the covariance formula

\[ \text{Cov}[X, Y] = \theta \left( e^{\alpha_x + \alpha_y} \cdot \sum_{m=0}^\infty \frac{1}{(2m)!} \sum_{j=0}^{2m} \left[ \frac{\beta_x}{\lambda_x^{\gamma_x}} \right]^m \left[ \frac{\beta_y}{\lambda_y^{\gamma_y}} \right]^{2m-j} \Gamma \left( 1 + \frac{j}{\gamma_x} + \frac{2m-j}{\gamma_y} \right) - \mu_x \mu_y \right). \]

Consider the interesting special case of the log-Laplace margins obtained setting \( \gamma_x = \gamma_y = 1 \), for which this simplifies to

\[ \text{Cov}[X, Y] = \theta \left( \frac{2 \cdot e^{\alpha_x + \alpha_y}}{2 - (\beta_x + \beta_y)} - \mu_x \mu_y \right). \]  

(D.35)

In the special case of equal coefficients of variation, whose squares are given by

\[ k_x^2 = \frac{1}{2} \left( 2 - \beta_x^2 \right)^2 - 1 = k_y^2 = \frac{1}{2} \left( 2 - \beta_y^2 \right)^2 - 1, \quad \beta_x = \beta_y = \sqrt{\frac{5}{2}}, \]  

(D.36)

one obtains using \( \mu_x = \frac{2}{2 - \beta_x^2} e^{\alpha_x}, \quad \mu_y = \frac{2}{2 - \beta_y^2} e^{\alpha_y} \), that \( \text{Cov}[X, Y] = \theta \cdot \sigma_x \sigma_y \), hence \( \theta = \rho_s \) coincides with Pearson’s \( \rho \) as in the Examples D.1 to D.3. Note that the last property also holds for the more general case \( \gamma_x = \gamma_y > 1 \), whose verification is left to the reader.
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